

A remark on scalar valued multiplicative functions of matrices.

To Professor Otto Varga on his 50th birthday.

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Let $K_n^\times$ denote the multiplicative semigroup of n -rowed square matrices over the real (or complex) field K . The mapping $A \rightarrow \varphi A$ of $K_n^\times$ into K is called *multiplicative*, if the equation

$$\varphi(AB) = \varphi A \varphi B, \quad A, B \in K_n^\times$$

holds. M. KUCHARZEWSKI [5] has proved that every mapping $A \rightarrow \varphi A$ of this form is a multiplicative function (in the usual sense) of $\det A$. M. KUCZMA [6] has simplified the proof of M. KUCHARZEWSKI's theorem. The object of the present paper is to prove this theorem in another way.

We shall use the well known theorem [4] that every matrix A has a factorization $A = HU$, where H is Hermitian and U is unitary, hence both factors are equivalent to diagonal matrices. On the other hand, the value of φ is the same for equivalent matrices, just as the value of the determinant, since

$$\varphi A = \varphi(BB^{-1}A) = \varphi B(\varphi B^{-1})\varphi A = \varphi B\varphi A\varphi B^{-1} = \varphi(BAB^{-1}).$$

So A is a product of two function values depending on diagonal matrices, hence it depends only on a diagonal matrix D having the same determinant as A since also the determinant is a multiplicative function. Therefore, considering the factorization

$$\begin{aligned} D &= \begin{bmatrix} d_1 & 0 & \dots \\ 0 & d_2 & 0 & \dots \\ 0 & 0 & d_3 & 0 & \dots \\ \dots & & & & \end{bmatrix} = \begin{bmatrix} d_1 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots \\ 0 & d_2 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & & & & \end{bmatrix} \dots = \\ &= \prod_{k=1}^n P_k \begin{bmatrix} d_k & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & & & & \end{bmatrix} P_k^{-1}, \end{aligned}$$

where $\mathbf{P}_k$ consists of the elements of the unit matrix, but the first and k th rows are permuted, we get

$$\begin{aligned}\varphi \mathbf{A} = \varphi \mathbf{D} &= \prod_{k=1}^n \varphi \begin{bmatrix} d_k & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{bmatrix} = \varphi \begin{bmatrix} \prod_{k=1}^n d_k & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{bmatrix} = \\ &= f(\det \mathbf{D}) = f(\det \mathbf{A})\end{aligned}$$

for every $\mathbf{A} \in K_n^\times$.

The theorem proved above gives a possibility of axiomatizing determinants without coordinates¹⁾ by the multiplicativity and by the homogeneity:

$$\varphi(\lambda \mathbf{A}) = \lambda^n \varphi \mathbf{A}, \quad \lambda \in K, \mathbf{A} \in K_n^\times,$$

e. g., if K is the real field and n is odd.²⁾

As a corollary we get as characteristic properties of the determinant the multiplicativity and the additivity for a column and row vector, respectively. These properties were used by M. STOJAKOVIČ [7] to characterize the determinant.

Bibliography.

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¹⁾ The problem of characterizing determinants without coordinates has arisen in [1–2].

²⁾ J. GÁSPÁR [3] could characterize Dieudonné's determinant over a field by the multiplicativity and by the homogeneity.