

A comment on E. Fried's Galois modules

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In [2], E. FRIED examines the following situation: K is a field, Γ is a finite-dimensional algebra over K and L is a finitely generated right Γ -module satisfying the following conditions

- (1) If $M = \text{Hom}_K(L, L)$ is the endomorphism ring of L , Then $\Gamma M = M$.
- (2) If J is any right ideal of Γ , $d(JM)/d(M) = d(J)/d(\Gamma)$,

where d denotes dimension over K . The aim of this note is to show (in Theorem 2 and its corollaries) that these modules L can be characterized as special injective modules which are uniquely determined by their dimensions, and that $L = \Gamma$ satisfies (1) and (2) if and only if Γ is a Frobenius algebra (cf. [1]).

We shall assume throughout that Γ is an algebra with unit and that all modules are unitary, so that (1) is automatic. It is probably possible to avoid this assumption by judicious adjunction of a unit.

If M is any Γ -module and m is a nonnegative integer, we shall use mM to denote the direct sum of m isomorphic copies of M . In these terms we define a left Γ -module M to be *semifree* (called „regular” by NAKAYAMA [3]) if $mM \cong n\Gamma$ for some nonzero m and n ($\cong$ means isomorphism of left Γ -modules). If $m = 1$, M is free. We shall call a left Γ -module M a *Fried module* if M satisfies (2). The following two lemmas are then obvious:

Lemma 1. *The left Γ -module Γ is both semifree and a Fried module.*

Lemma 2. *If $mM \cong nN$ then M is a Fried module (resp. semifree) if and only if N is a Fried module (resp. semifree).*

Theorem 1. *M is a Fried module if and only if M is semifree.*

PROOF. If M is semifree, then $mM \cong n\Gamma$ and the fact that M is a Fried module follows immediately from Lemmas 1 and 2.

Conversely, consider first the case where Γ is a simple algebra. Then Γ has a unique simple (= irreducible) left module V and every finitely generated Γ -module is isomorphic to some nV . Thus every finitely generated module is semifree, by Lemmas 1 and 2.

Next, if Γ is semisimple, write $\Gamma = J_1 \oplus \cdots \oplus J_k$ where each J_i is a simple algebra. Then $M = \bigoplus J_i M$ and, by the previous case, $m_i J_i M \cong n_i J_i$ for some m_i and n_i . But $d(\Gamma)d(J_i M) = d(M)d(J_i)$ since M is a Fried module. Thus $n_i/m_i = d(M)/d(\Gamma)$ is independent of i .

If m is the least common multiple of the m_i , write $n_i/m_i = n/m$; then $mM = \bigoplus m J_i M = \bigoplus (m/m_i) m_i J_i M \cong \bigoplus (m/m_i) n_i J_i = \bigoplus n J_i = n\Gamma$, so that M is semifree.

Finally, if Γ has a radical R , we consider the semisimple algebra Γ/R . If M is a Fried module over Γ , then M/RM is a Fried module over Γ/R , for if J/R is any right ideal in Γ/R (J a right ideal in Γ containing R), then

$$\begin{aligned} d([J/R][M/RM])/d(M/RM) &= d(JM/RM)/d(M/RM) = \\ &= [d(JM) - d(RM)]/[d(M) - d(RM)] = \\ &= [d(JM)/d(M) - d(RM)/d(M)]/[1 - d(RM)/d(M)] = \\ &= [d(J)/d(\Gamma) - d(R)/d(\Gamma)]/[1 - d(R)/d(\Gamma)] = d(J/R)/d(\Gamma/R). \end{aligned}$$

Thus M/RM is semifree; say $m(M/RM) \cong n(\Gamma/R)$. Let $M' = mM$ and $N = n\Gamma$. Then $N/RN \cong M'/RM'$. This gives a Γ -homomorphism α of N to M'/LM' . Since N is free, α can be lifted to a homomorphism α' of N to M' . Then $\alpha'(N) + RM' = M'$, whence, by usual techniques, $\alpha'(N) + R^k M' = M'$, $\alpha(N) = M'$. Since M' is a Fried module,

$$d(M'/RM') = d(M') - d(RM') = d(M')(1 - d(R)/d(\Gamma)) = d(M')d(\Gamma/R)/d(\Gamma).$$

Similarly,

$$d(N)d(\Gamma/R)/d(\Gamma) = d(N/RN) = d(M'/RM') = d(M')d(\Gamma/R)/d(\Gamma),$$

which proves $d(N) = d(M')$. This, paired with $\alpha'(N) = M'$ shows that α' is an isomorphism, $mM = M' \cong N = n\Gamma$, M is semifree.

Corollary. *If M and N are Fried modules, then M is isomorphic to N if and only if $d(M) = d(N)$.*

PROOF. Write M , N and Γ as direct sums of indecomposable modules: $M = \bigoplus m_i I_i$, $N = \bigoplus n_i I_i$, $\Gamma = \bigoplus c_i I_i$. Since $mM \cong m'\Gamma$ and $nN \cong n'\Gamma$ by Theorem 1, we have $mm_i = m'c_i$ and $nn_i = n'c_i$ by the Krull—Schmidt—Remak theorem. Thus $m_i = (m'/m)c_i$ and $n_i = (n'/n)c_i$. But $d(M) = d(N)$ implies $m'/m = n'/n$, so $m_i = n_i$ and $M \cong N$.

Now to return to FRIED's context, if L is a right Γ -module and $M = \text{Hom}_K(L, L)$, let L' be the dual $\text{Hom}_K(L, K)$ of L . The transposes of elements of M act on L' in the usual way, making L' a left M -module and hence also a left Γ -module. Moreover, the left $(M-$, hence $\Gamma-$) module M is isomorphic to a direct sum of copies of L' , so M is a Fried module if

and only if L' is. Translating Theorem 1 into conditions on dual modules, we have

Theorem 2. *The following are equivalent*

- (3) *A right Γ -module L satisfies (1) and (2).*
- (4) *The dual $L' = \text{Hom}_K(L, K)$ of L is a semifree left Γ -module, i. e., $mL' \cong n\Gamma$ as left Γ -modules.*
- (5) *There exist nonzero integers m and n such that $mL \cong n\text{Hom}_K(\Gamma, K)$ as right Γ -modules.*
- (6) *If $L = \bigoplus l_i I_i$ and $\text{Hom}_K(\Gamma, K) = \bigoplus c'_i I_i$ with nonisomorphic, indecomposable I_i , then $l_i/l_j = c'_i/c'_j$ for all i, j .*

Corollary 1. *If two right Γ -modules L_1 and L_2 satisfy the conditions of Theorem 2, then $L_1 \cong L_2$ if and only if $d(L_1) = d(L_2)$.*

This Corollary is proved exactly as is the Corollary to Theorem 1; or it can be reduced to that Corollary by taking duals.

Corollary 2. *The right Γ -module Γ satisfies the conditions of Theorem 2 if and only if Γ is a Frobenius algebra; i. e., the right module Γ (which gives the right regular representation of Γ) is isomorphic to $\text{Hom}_K(\Gamma, K)$ (which gives the left regular representation).*

PROOF. The „if” part is clear from Theorem 2, (5). The „only if” follows from Corollary 1.

Thus the major theorem of [2] (Theorem 7) can be translated into the assertion that if Γ is a Frobenius algebra, if a right Γ -module L is the dual of a semifree left module, and if $d(L) = d(\Gamma)$, then $L \cong \Gamma$. The proof of the normal basis theorem then consists in showing that the group algebra is a Frobenius algebra [2, Theorem 9] (cf. also [1]), that the field L is the dual of a semifree module over the group algebra of the Galois group [1, Theorem 10] and that $d(L) = d(\Gamma)$.

Bibliography

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