

Connected ordered topological groupoids with idempotent endpoints¹⁾

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The groupoids described in this title will be called g -threads. More specifically, a g -thread is a system $G(\circ, <)$ with the following properties:

- (1) G is a groupoid with respect to $\circ$, i. e., $\circ$ is a single valued binary operation defined in G for every pair of elements $a, b \in G$.
- (2) $<$ is a total (i. e., linear or simple) order relation in G .
- (3) The mapping $p(x, y) = x \circ y$ of $G \times G$ into G is continuous in the order topology.
- (4) G is connected in the order topology.
- (5) G has a zero, 0 , and a unit, u [3]. Zero is the least element, and u is the greatest element of G with respect to $<$.

CLIFFORD has considered g -threads obeying the associative law. He suggested the above terminology. We will denote $a \circ b$ by ab .

Our main result states: if G is a g -thread whose elements with unique powers form a dense subset and if G satisfies the cancellation law then there exists a function F from G onto the unit interval $[0, 1]$ of real numbers under the usual topology and the usual multiplication, which is an isomorphism as well as order preserving homomorphism. In particular any continuous multiplication on the interval $[0, 1]$ with the usual topology is the usual multiplication if it satisfies the above conditions.

Definition 1. A groupoid with a zero is said to satisfy the *cancellation law* if and only if $xc = yc$ or $cx = cy$ implies $x = y$ for all $x, y, c \in G$ with $c \neq 0$.

Lemma 1. *Let G be a g -thread satisfying the cancellation law. Then $x < y$ implies $xc < yc$ and $cx < cy$ for all $x, y, c \in G$ with $c \neq 0$.*

¹⁾ This is essentially a condensation of a part of a doctoral dissertation written at the University of Tennessee under the direction of Professor O. G. HARROLD, Jr.

PROOF. Let $x, y \in G$ such that $x < y$. Let $A = \{d: xc < yc \text{ for all } c \in [d, u]\}$. Then, $A \neq \emptyset$ since $u \in A$. Let $k = \inf. \{d: d \in A\}$. We suppose $k > 0$. We first show that $xk < yk$. If $xk = yk$, then $x = y$. Suppose $xk > yk$. Then there exists a neighborhood $V(k)$ of k and a $d \in V(k) \cap A$ such that $xd > yd$. This contradicts the fact that $d \in A$. Hence, $xk < yk$. Thus, one may find an open interval $U(k)$ about k such that $xc < yc$ for all c in $U(k)$. Since there is a $d \in U(k) \cap A$, $xc < yc$ for all $c \in [d, u]$. But, there exists a $t \in U(k)$ such that $t < k$. Hence $xc < yc$ for all $c \in [t, u]$ and $t \in A$. This contradicts the definition of k . Hence, $k = 0$. Thus, $xc < yc$ for all $c \neq 0$. A similar argument yields $cx < cy$ for $c \neq 0$.

Definition 2. An element $a \in G$ is said to be *power associative* if and only if all the powers of a are unique.

Lemma 2. *If G is a g -thread which satisfies the cancellation law, then*

- (1) $xy \leq \min(x, y)$
 - (2) if $x \leq y$ and $w \leq v$, then $xw \leq yv$.
 - (3) if $x \neq u$ is power associative, $(x^n: n \in I)$ converges to 0.
- (I will denote the positive integers.)

PROOF. (1) and (2) follow immediately from Lemma 1. (3) Since x is power associative, $u \geq x \geq x^2 \geq x^3 \dots$ by Lemma 1. Since x is power associative $(x^n: n \in I)$ is a semigroup and $(x^n: n \in I)$ must cluster at an idempotent [3]. By the cancellation law, 0 and u are the only idempotents of G . Since $x \neq u$, $(x^n: n \in I)$ converges to 0.

Lemma 3. *Let G be a g -thread which satisfies the cancellation law and in which the power associative elements form a dense subset. Then, G is an arc and G is an abelian topological semi-group.*

PROOF. Clearly, each $a \in G$ is power associative. Let $\mathcal{A}^n$ be the diagonal of the cartesian product of G , n times. For each n define $f_n: \mathcal{A}^n \rightarrow G$ by $f_n(y) = x^n$ where x is the projection of y onto G . Since $f_n(\mathcal{A}^n) = G$, every $x \in G$ has an n^{th} root. If $x \neq 0$, x has a unique square root by virtue of Lemma 1. By induction, x has a unique 2^m -th root. If $x, y \in G$ and $x < y$, $x^2 < y^2$ by Lemma 1. It follows $x^{2^m} < y^{2^m}$ for all positive integers m . We call this result (1).

Let us define for $x \in G$, $x \neq 0$

$$x^{p/2^q} = (x^{1/2^q})^p$$

where p and q are positive integers. It is immediate from the fact that the 2^q -th roots are unique, that x^r is well defined for any positive dyadic rational

r , regardless of the representation of r . Clearly,

$$(2) \quad x^r x^s = x^{r+s}.$$

Let r and s be dyadic rationals such that $r < s$. Then, there exists a dyadic rational t such that $r + t = s$. Hence $x^s = x^r x^t \leq x^r$ by virtue of (2) and Lemma 2. Therefore, if (r_n) is a monotone increasing sequence of dyadic rationals, (x^{r_n}) is a monotone decreasing sequence in G . For $x \in G$, $x \neq 0$ or u , let $D = (x^r: r \text{ is a positive dyadic rational})$. By (2), D is an abelian submob of G . We claim $\bar{D} = G$. Let $V(0)$ be any neighborhood of zero. Since $x \neq u$, $x^n \rightarrow 0$ by Lemma 2. Thus $V(0) \cap D \neq \emptyset$. Let $V(u)$ be any basis element containing u . Let $y \in V(u)$ with $y < u$. Clearly $y^n \rightarrow 0$. Thus there exists a positive integer n such that $y^{2^n} < x$. Hence, by (1) $y < x^{1/2^n}$ and $V(u) \cap D \neq \emptyset$. Let t be an arbitrary element of G , $t \neq 0$ and $t \neq u$ and let $B = (s: a < s < b)$ be a basis element in the order topology containing t . Without any loss of generality, we may assume $0 < a < b < u$. Let

$$R' = (r: x^r \geq b, r \text{ a positive dyadic rational})$$

$$R'' = (r: x^r \leq a, r \text{ a positive dyadic rational}).$$

Let r' be the l. u. b. of R' and r'' be the g. l. b. of R'' . Clearly, $r' \leq r''$. If $r' < r''$, D meets B .

Therefore, we assume $r' = r''$. Let (p_n) be a monotone increasing sequence of dyadic rationals of R' converging to r' . Then (x^{p_n}) is a monotone decreasing sequence converging to its g. l. b. $c \geq b$. Let W be an open set containing c so that $W \cap [0, a] = \emptyset$. There exists open sets U and V with $u \in U$ and $c \in V$ such that $UV \subset W$. Since $D \cap U \neq \emptyset$, $x^r \in U$ for some r . There exists a p_k in the sequence (p_n) such that $x^{p_k} \in V$ and $r + p_k > r' = r''$. Hence $x^{r+p_k} \in [0, a] \cap W$. From this contradiction, it follows that $\bar{D} = G$. Since the closure of an abelian subsemi-group of a topological groupoid is an abelian semi-group, G is an abelian topological semi-group. Since G is separable, G is an arc by virtue of a theorem of WILDER ([4], pp. 21–39).

We next state a result due to FAUCETT [2].

Lemma 4. *Let S be a compact semi-group with a zero and a unit and no other idempotents. Suppose S is irreducibly connected between 0 and u . (2). Assume further that S contains no non-zero algebraic nilpotents (2). Then there exists a function F from S onto the unit interval $[0, 1]$ of real numbers with the usual topology and the usual multiplication, that is an isomorphism as well as an order preserving homeomorphism.*

If G is a groupoid with a zero such that G satisfies the cancellation law and every element of G is power associative, then it is easy to see that $x \in G$ and $x^n = 0$ for some n implies $x = 0$.

Therefore by virtue of Lemma 4 we have:

Theorem. *Let G be a g -thread such that G satisfies the cancellation law and the power associative elements of G form a dense subset. Then there exists a function f from G onto the unit interval $[0, 1]$ of real numbers under the usual topology and the usual multiplication which is an isomorphism as well as an order preserving homeomorphism.*

We give an example to indicate the effect of the topology on the algebraic structures.

Example 1. An example of a finite groupoid with a zero and a unit which obeys the cancellation law and in which every element has unique powers but which is non-associative and non-abelian is given by $G = (0, 1, 2, 3, 4, 5)$ with the following multiplication table:

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	1	5	3	4
3	0	3	4	1	5	2
4	0	4	5	2	1	3
5	0	5	3	4	2	1

We next give an example that seems to indicate the necessity of requiring certain elements to be power associative.

Example 2. An example of a continuous multiplication on $[0, 1]$ with the usual topology which is non-associative and non-abelian, which has 0 as a zero, which has 1 as a right unit, and which obeys the cancellation law. We define: $a \circ b = ab^2$ where the last multiplication is that of real numbers. Zero and 1 are the only power associative elements.

Bibliography

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