

A note on entire functions

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1. Let $f(z)$ be an entire function of order ρ ($0 < \rho < \infty$); let $\mu(r, f)$ be the maximum term of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| = r$ and let as usual $M(r, f)$ and $M(r, f')$ denote the maximum moduli of $f(z)$ and $f'(z)$ on $|z| = r$. G. VALIRON has proved the following results:

$$(1) \quad M(r, f') < r^{\frac{3}{2}\rho-1+\varepsilon} \mu(r, f)$$

for all sufficiently large values of r exterior to a set of segments in which the total variation of $\log r$ is finite.

$$(2) \quad M(r, f') < r^{2\rho-1+\varepsilon} \mu(r, f) \quad \text{for all } r \geq r_0$$

(see G. Valiron [1], p. 169, 212).

In this note we prove that (1) is a best possible result in the sense that $(3/2)\rho$ cannot be replaced by a smaller number. Further we provide a simple and alternative proof of (2). We also prove:

$$(3) \quad v(r, f) \mu(r, f) \leq r M(r, f')$$

where $v(r, f)$ is the rank of the maximum term.

2. We now establish (1).

Consider

$$f(z) = \cos \sqrt{z}.$$

Then $f(z)$ is an entire function of order $\rho = 1/2$. Let $R_n = (2n-1)2n$. Then for $R_n \leq r < R_{n+1}$

$$\mu(r, f) = \frac{r^n}{(2n)!} \sim \frac{r^n}{(2n)^{2n+1/2} e^{-2n} \sqrt{2\pi}} \sim \frac{e^{\sqrt{r}}}{\sqrt{2\pi} r^{1/4}}.$$

Now
$$f'(z) = -1/2 \left(1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \dots \right),$$

so $|f'(-r)| = M(r, f') = 1/2 \left(1 + \frac{r}{3!} + \frac{r^2}{5!} + \dots \right) = \frac{e^{\sqrt{r}} - e^{-\sqrt{r}}}{4\sqrt{r}}.$

Hence

$$\frac{M(r, f')}{\mu(r, f)} \sim \frac{e^{\sqrt{r}} - e^{-\sqrt{r}}}{4\sqrt{r}} \frac{\sqrt{2\pi} r^{\frac{1}{4}}}{e^{\sqrt{r}}} = \frac{\sqrt{2\pi}}{4} \left\{ 1 - \frac{1}{e^r} \right\} r^{\frac{3}{2}q-1}$$

and the result follows.

Proof of (2):

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n.$

Then

$$r M(r, f') \leq \sum_{n=1}^{\infty} n |a_n| r^n.$$

Let $r = \left(\frac{n}{\varrho + \varepsilon} \right)^{\frac{1}{\varrho + \varepsilon}}.$

Then by CAUCHY's inequality

$$|a_n| r^n \leq M(r, f) < e^{r^{\varrho + \varepsilon}} = e^{\frac{n}{\varrho + \varepsilon}} = \left\{ \frac{e(\varrho + \varepsilon)}{n} r^{\varrho + \varepsilon} \right\}^{\frac{n}{\varrho + \varepsilon}} \quad (r \geq r_0).$$

Put $N = [2e(\varrho + \varepsilon)r^{\varrho + \varepsilon}]$ where $[x]$ denotes the integral part of x .

Then

$$\begin{aligned} r M(r, f') &\leq \sum_{n=1}^N n |a_n| r^n + \sum_{N+1}^{\infty} n |a_n| r^n \leq \\ &\leq \mu(r, f) \frac{N(N+1)}{2} + \sum_{N+1}^{\infty} n \left\{ \frac{e(\varrho + \varepsilon)}{n} r^{\varrho + \varepsilon} \right\}^{\frac{n}{\varrho + \varepsilon}} \leq \\ &\leq \mu(r, f) \frac{N(N+1)}{2} + \sum_{N+1}^{\infty} n \left\{ \frac{e(\varrho + \varepsilon)}{N} r^{\varrho + \varepsilon} \right\}^{\frac{n}{\varrho + \varepsilon}} \leq \\ &\leq \mu(r, f) \frac{N(N+1)}{2} + \sum_{N+1}^{\infty} n 2^{\frac{-n}{\varrho + \varepsilon}}. \end{aligned}$$

Now for $n \geq n_0$

$$\sum_{N+1}^{\infty} n 2^{\frac{-n}{\varrho + \varepsilon}} < \sum_N^{\infty} 2^{\frac{-n}{2(\varrho + \varepsilon)}} = O(1)$$

and the result follows.

To prove (3) we observe that

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Hence by CAUCHY's inequality we have

$$(4) \quad M(r, f') \geq n |a_n| r^{n-1}.$$

If we set $R_n = \left| \frac{a_{n-1}}{a_n} \right|$ then for $R_n \leq r < R_{n+1}$, we have

$$\begin{aligned} \mu(r, f) &= |a_n| r^n, \\ \nu(r, f) &= n. \end{aligned}$$

Hence from (4) we have

$$M(r, f') \geq \nu(r, f) \frac{\mu(r, f)}{r}.$$

This proves (3).

Bibliography

- [1] G. VALIRON, *Fonctions Analytiques*, Paris, 1954.

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