New proof of a theorem on decimal periodicity[1])

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1. Introduction. The subject of periodic decimal fractions is indeed a very old one. As a topic, it was deemed importart enough by DICKSON so that he devoted a chapter to its discussion in his *History* [2]; and one may there see the role it has played in the development of other mathematical notions.

Let us first recall the basic terminology of the subject. The decimal representation of the (rational) number x is called *periodic* if the digits eventually repeat, i. e., if there exist integers $r \geqq 0$ and $s > 0$ for which

$$10^r x - [10^r x] = \sum_{i=1}^{\infty} \frac{a_i}{10^i} \tag{1}$$

and

$$a_{s+i} = a_i \qquad i = 1, 2, \ldots \tag{2}$$

for the decimal digits a_i (as usual, the square brackets in (1) refer to the „greatest integer" function). If s is the least positive integer for which (2) is true, then the sequence $a_1 a_2 \ldots a_s$, obtained from (1), is called the *period* of x. This least number s is called the *length* of the period; alternately, we also refer to s as the *periodicity* of x.

It was learned by the early investigators that the basic problem of determining the periodicity of a decimal fraction a/b, in lowest terms, is quickly reduced to the determination of the periodicity of $1/p$ for each prime divisor p of b. (See [1] and [2]).

Of course, to each prime p there corresponds a unique positive integer s: the periodicity of $1/p$. It is natural to ask if *each* positive integer is used in this correspondence; that is, given any positive integer s, is there a prime p such that $1/p$ has periodicity s?

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[1]) Presented to the American Mathematical Society Septemper 4, 1959.

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The answer to this (and more) was first given by BANG in his investigations of a function devised by SYLVESTER. BANG's result was given later by DICKSON and was generalized in a paper by BIRKHOFF and VANDIVER. (References to these papers may be found by consulting [2, Chapter XVI].) The methods are algebraic, involving theorems about congruences and cyclotomic polynomials with corresponding reference to roots of unity, group theoretic notions, etc.

We give here an answer by completely elementary notions, using nothing but inequalities and logarithms; and since the notions are completely elementary, we have kept the language of the paper completely elementary as well. It is interesting that what amount to rather crude arithmetic estimates is enough to obtain the result.

For the sake of completeness, we prove in § 2 a divisibility theorem (a property known to JOHN WALLIS [3]) which gives a criterion for finding the periodicity of the reciprocal of a prime. In § 3, we give by use of this criterion a new proof that each positive integer qualifies as the length of the decimal period of the reciprocal of some prime.

Partly from respect for the history of the subject, this paper deals with decimal properties only. It is clear that the basic notions hold for bases other than 10 and that the proof of the theorem of § 3 can be modified to cover a certain few other bases. The stronger machinery of the aforementioned algebraic methods seems necessary, however, to prove the correspondingly stronger result to the effect that the base is irrelevant with the exceptions that when base 2 is used, no prime p has the property that $1/p$ has periodicity 6; and when any base 2^k-1 is used, no prime p has the property that $1/p$ has periodicity 2. The point is that in the algebraic arguments of BANG *et al*, the number 10 plays no role, whereas 10 does play a very important part in our ability to make the needed estimates.

2. Divisibility and periodicity. If x is the reciprocal of a prime other than 2 or 5, then the integer r in (1) can always be chosen to be 0; and so for reciprocals of such primes p, we may write (1) as

$$\frac{1}{p} \sum_{i=1}^{\infty} \frac{a_i}{10^i}. \tag{3}$$

Now supposing s to be the length of the period of this number $1/p$, so that (2) holds for the digits a_i of (3), we have that

$$\frac{10^{ks}-1}{p} = \sum_{i=1}^{ks} a_i \cdot 10^{ks-i},$$

an integer. On the other hand, it is clear that for any integer j with $0<j<s$, the number $(10^{ks+j}-1)/p$ is not an integer. We have proved, then, the following

Proposition. *The positive integer* 10^m-1 *is divisible by* p, *a prime distinct from* 2 *and* 5, *if and only if* s *divides* m, *where* s *is the decimal periodicity of* $1/p$.

This proposition thus gives a divisibility criterion for determining the periodicity of the reciprocal of a prime p: the length of the decimal period of $1/p$ is s if p divides 10^s-1 but does not divide any of the numbers 10^1-1, $10^2-1, \ldots, 10^{s-1}-1$. For example, $10^1-1=3^2$, $10^2-1=3^2\cdot 11$, $10^3-1=3^3\cdot 37$, and $10^4-1=3^2\cdot 11\cdot 101$; therefore 1/3, 1/11, 1/37 and 1/101 have periodicities 1, 2, 3 and 4, respectively.

This means that if for each positive integer s, we always obtain a prime factor of 10^s-1 which has never occurred in the factorization into primes of any of the numbers 10^k-1 where $1\leqq k\leqq s-1$, then each positive integer s qualifies as the decimal periodicity of the reciprocal of some prime.

3. The divisibility theorem. The pricipal result of this paper is a new proof of the

Theorem. *For each positive integer* n, *the number* 10^n-1 *is divisible by a prime which does not divide any of the numbers* 10^k-1 *where* k *is a positive integer less than* n.

PROOF. The theorem is trivial for $n=1$, and we proceed by first considering the cases in which n is (1) a prime, (2) the product of two primes, and (3) the product of three primes. Covering these three special cases is necessary for the final argument in which n is (4) the product of at least four primes.

Case 1: $n=p_1$, *a prime.* Let p be a prime divisor of $10^{p_1}-1$. By the Proposition of § 2, p_1 must be a multiple of the periodicity of $1/p$; and since p_1 is a prime, this means that p can occur as a prime factor of 10^k-1 where $0<k<p_1$ only in the case $k=1$. But $10^1-1=3^2$, while $10^{p_1}-1=99\ldots9$ (p_1 digits). Dividing $10^{p_1}-1$ by 9, we obtain $11\ldots1$ (p_1 digits), and we shall show that this number has a prime factor other than 3. It is well known that if 3 divides $11\ldots1$ (p_1 digits), then the sum of these p_1 digits must be a multiple of 3. Hence, p_1 is a multiple of 3; and since p_1 is a prime, $p_1=3$. It remains only to check that 10^3-1 is not a power of 3, and this is not so since $10^3-1=3^3\cdot 37$.

Case 2: $n=p_1p_2$, *the product of two primes.* Again by the Proposition of § 2, note that we need only show $10^{p_1p_2}-1$ to be divisible by a prime

which divides neither $10^{p_1}-1$ nor $10^{p_2}-1$. Now notice that

$$10^{p_1 p_2}-1 = \{1+(10^{p_1}-1)\}^{p_2}-1$$

$$(4) \qquad = p_2(10^{p_1}-1)+\frac{p_2(p_2-1)}{2!}(10^{p_1}-1)^2+\cdots+(10^{p_1}-1)^{p_2}.$$

A close examination of (4) shows that a prime factor of $10^{p_1}-1$ cannot divide $10^{p_1 p_2}-1$ any more often than it divides $p_2(10^{p_1}-1)$. And by interchanging p_1 and p_2 in (4), we see through symmetry that a prime factor of $10^{p_2}-1$ cannot divide $10^{p_1 p_2}-1$ any more than it divides $p_1(10^{p_2}-1)$. Now if $p_1 = p_2 = 2$, it is simple to check that the theorem is true; so we assume $p_1 \neq 2$ or $p_2 \neq 2$. With this assumption, it follows from Lemma 1 below that

$$(5) \qquad p_1+p_2+\log(p_1 p_2)<p_1 p_2,$$

where here and in the sequel all logarithms are taken base 10. By (5), we obtain

$$(6) \qquad p_1 p_2 \cdot 10^{p_1+p_2}<10^{p_1 p_2},$$

and since

$$10^{p_1+p_2}>10^{p_1+p_2}-10^{p_1}-10^{p_2}+1,$$

we have as a result of (6) that

$$(7) \qquad p_1 p_2(10^{p_1}-1)(10^{p_2}-1)<10^{p_1 p_2}-1.$$

This inequality (a special case of „the idea" of our proof) assures us that $10^{p_1 p_2}-1$ must contain a prime factor which divides neither $10^{p_1}-1$ nor $10^{p_2}-1$, and we now need only establish the lemma which gives (5).

Lemma 1. *If a and b are integers with $a \geqq 2$ and $b \geqq 3$, then $a+b+\log(ab)<ab$.*

PROOF Induction: note that $2+3+\log 6<6$ and that if $a+b+\log(ab)<ab$, then

$$\begin{aligned} a+1+b+\log(ab+b) &= a+1+b+\log(ab+b)+\log(ab)-\log(ab) \\ &< ab+1+\log\left(\frac{a+1}{a}\right) \\ &< ab+2 \\ &< (a+1)b. \end{aligned}$$

Case 3: $n = p_1 p_2 p_3$, the product of three primes. As before, we need only show that $10^{p_1 p_2 p_3}-1$ contains a prime factor which is not a factor

of $10^{p_1p_2}-1$, $10^{p_2p_3}-1$, or $10^{p_1p_3}-1$. An expression similar to (4) can be written to show that a prime divisor of $10^{p_1p_2}-1$ cannot divide $10^{p_1p_2p_3}-1$ any more often than it divides $p_3(10^{p_1p_2}-1)$. By symmetry, the preceding sentence is true for each of the other two combinations of subscripts, and so the theorem will be verified for this case if we can establish the inequality

$$p_3(10^{p_1p_2}-1)\cdot p_1(10^{p_2p_3}-1)\cdot p_2(10^{p_1p_3}-1)<10^{p_1p_2p_3}-1, \tag{8}$$

the analog of (7). To obtain (8), we will use a lemma parallel in statement and purpose to Lemma 1.

Lemma 2. *If a, b and c are integers with $a\geqq 2$, $b\geqq 3$, $c\geqq 8$, then $ab+bc+ac+\log(abc)<abc$.*

PROOF. Proof is again by induction. First, $6+24+16+\log 48<48$. Next, if $ab+bc+ac+\log(abc)<abc$, then similar to the proof of Lemma 1, we have

$$\begin{aligned}(a+1)b+bc+(a+1)c+\log(a+1)bc &< abc+b+c+\log\left(\frac{a+1}{a}\right)\\ &< abc+b+c+1\\ &< abc+bc=(a+1)bc,\end{aligned}$$

and the lemma is proved.

And now if p_1, p_2, p_3 is a set of three primes satisfying the hypothesis of Lemma 2, we obtain

$$p_1p_2+p_2p_3+p_1p_3+\log(p_1p_2p_3)<p_1p_2p_3.$$

Thus

$$p_1p_2p_3\cdot 10^{p_1p_2+p_2p_3+p_1p_3}<10^{p_1p_2p_3},$$

from which it follows that

$$(p_3\cdot 10^{p_1p_2})(p_1\cdot 10^{p_2p_3})(p_2\cdot 10^{p_1p_3})<10^{p_1p_2p_3},$$

and therefore (8) is true.

This leaves only the job of checking those cases in which the three primes do not satisfy the hypothesis of Lemma 2. These cases occur when n is of the form $4p_3$ (i. e., when $p_1=p_2=2$) or when n is any of the numbers 18, 27, 30, 42, 45, 50, 63, 70, 75, 98, 105, 125, 147, 175, 245, 343. In all but four of these cases (namely, when n is 30, 42, 70, or 105), a prime factor is repeated; and when a prime factor repeats, we need only establish a variant of (8):

$$p_1(10^{p_2p_3}-1)\cdot p_2(10^{p_1p_3}-1)<10^{p_1p_2p_3}-1. \tag{9}$$

This is because two of the factors on the left hand side of (8) will be identical (the inequality (9) being obtained by assuming either $p_2 = p_3$ or $p_1 = p_3$). Application of Lemma 1 gives (9). A rephrasing of Lemma 2 is possible when n is 70 or 105, the crucial inequalities being

$$10+35+14+\log 70<70$$
$$15+35+21+\log 105<105,$$

and so (8) is true for these cases. If n is 30 or 42, then (8) is not true, but the fractions 1/211 and 1/127 have these respective periodicities; and 211 and 127 are both primes.

Case 4: n is the product of at least four primes. As in the preceding cases, the product

$$p_1(10^{n/p_1}-1)p_2(10^{n/p_2}-1)\dots p_\lambda(10^{n/p_\lambda}-1), \tag{10}$$

where $n = p_1p_2\dots p_\lambda$ is the factorization of n into primes, contains all the possible prime factors of 10^n-1 which have arisen in the factorization of 10^k-1 with $k = 1, 2, \dots, n-1$. Indeed, the product (10) contains twice those primes which have arisen in the factorizations of $10^{n/p_ip_j}-1$, and so it will be sufficient to show, as in (7) and (8), that the inequality

$$\frac{\prod_{i=1}^{\lambda} p_i(10^{n/p_i}-1)}{\prod_{i<j}(10^{n/p_ip_j}-1)} < 10^n-1 \tag{11}$$

is true. (Actually, an alternate form of (8) along these lines is possible.)

The inequality (11) is implied by

$$\frac{\prod_{i=1}^{\lambda} p_i\cdot 10^{n/p_i}}{\prod_{i<j}(10^{n/p_ip_j}-1)} < 10^n.$$

This inequality may be expressed in the equivalent form

$$\log n+\sum_{i=1}^{\lambda}\frac{n}{p_i}-\sum_{i<j}\log(10^{n/p_ip_j}-1)<n, \tag{12}$$

and (12) may be written

$$\log n+\sum_{i=1}^{\lambda}\frac{n}{p_i}-\sum_{i<j}\frac{n}{p_ip_j}-\sum_{i<j}\log\left(\frac{10^{n/p_ip_j}-1}{10^{n/p_ip_j}}\right)<n. \tag{13}$$

Since

$$-\sum_{i<j}\log\left(\frac{10^{n/p_ip_j}-1}{10^{n/p_ip_j}}\right)=\sum_{i<j}\log\left(1+\frac{1}{10^{n/p_ip_j}-1}\right)$$

$$<\sum_{i<j}\frac{1}{2(10^{n/p_ip_j}-1)}$$

$$<\frac{1}{2}\sum\frac{10}{10^{n/p_ip_j}},$$

the inequality (13) is implied by

$$\log n+\sum_{i=1}^{\lambda}\frac{n}{p_i}-\sum_{i<j}\frac{n}{p_ip_j}+\frac{1}{2}\sum_{i<j}\frac{10}{10^{n/p_ip_j}}<n. \tag{14}$$

Because $\lambda\geqq 4$, the integer n/p_ip_j is the product of at least two primes, and therefore $n/p_ip_j\geqq 4$. It follows that

$$\frac{1}{2}\sum_{i<j}\frac{10}{10^{n/p_ip_j}}<\frac{1}{2}\cdot\frac{1}{100}\sum_{k=1}^{\infty}\frac{1}{10^k}=\frac{1}{1800},$$

and so (14) is in turn implied by

$$\log n+\sum_{i=1}^{\lambda}\frac{n}{p_i}-\sum_{i<j}\frac{n}{p_ip_j}+\frac{1}{1800}<n$$

which we now write as

$$\frac{\log n}{n}+\sum_{i=1}^{\lambda}\frac{1}{p_i}-\sum_{i<j}\frac{1}{p_ip_j}+\frac{1}{1800n}<1.$$

This last inequality is equivalent to

$$\frac{\log n}{n}+\sum_{i=1}^{\lambda}\frac{1}{p_i}-\frac{1}{2}\left\{\sum_{i=1}^{\lambda}\frac{1}{p_i}\right\}^2+\frac{1}{2}\sum_{i=1}^{\lambda}\frac{1}{p_i^2}+\frac{1}{1800n}<1. \tag{15}$$

Now note that

$$\frac{1}{2}\sum_{i=1}^{\lambda}\frac{1}{p_i^2}\leqq\frac{1}{4}\sum_{i=1}^{\lambda}\frac{1}{p_i},$$

which means that (15) is implied by

$$\frac{\log n}{n}+\frac{5}{4}\sum_{i=1}^{\lambda}\frac{1}{p_i}-\frac{1}{2}\left\{\sum_{i=1}^{\lambda}\frac{1}{p_i}\right\}^2+\frac{1}{1800n}<1. \tag{16}$$

But since 2^4 is the smallest possible value for n, we have

$$\frac{\log n}{n}+\frac{1}{1800n}<\frac{7}{32}$$

making (16) implied by

$$-\frac{1}{2}\left\{\sum_{i=1}^{\lambda}\frac{1}{p_i}\right\}^2+\frac{5}{4}\sum_{i=1}^{\lambda}\frac{1}{p_i}-\frac{25}{32}<0. \tag{17}$$

We note that the left hand side of (17) is a quadratic polynomial, and it is easily seen that (17) holds whenever

$$\sum_{i=1}^{\lambda}\frac{1}{p_i}>\frac{5}{4},$$

which is clearly the case.

This completes the proof of Case 4 and of the Theorem.

Bibliography

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(Received May 20, 1961.)