On certain embeddability criteria for group amalgams

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1. Introduction

Corresponding to the free product, the direct product, and the various regular products of groups, intermediate between the free and direct products, that have been introduced by GOLOVIN, MORAN, and others, one can introduce the generalized free product, the generalized direct product, and the various generalized products that are intermediate between them: these generalized products are characterized by the fact that their *constituents* need not intersect trivially. The simplest case, and the only one we shall here be concerned with, is that of two constituent groups A and B intersecting in a subgroup H, and thus forming the *amalgam*

$$\mathfrak{A} = \mathrm{am}(A, B; H).$$

When the amalgamated subgroup H is not the trivial group, the generalized products need not exist; and one naturally asks for existence criteria.

Such existence criteria are known for the generalized free product, where a classical theorem of SCHREIER [3] says that the criterion is void, in other words, the generalized free product of an amalgam of two groups always exists; for the generalized direct product, where the necessary and sufficient condition for existence is simply that H is central in both A and B (see, for example, [2] and the literature there quoted); and for the generalized GOLOVIN second nilpotent (or GN_2) product, where the necessary and sufficient conditions are unexpectedly complicated, and expressed in terms of homomorphisms of certain subgroups of the tensor product of A and B into A and into B (see [4]). These last conditions may appear unnatural, and it would be greatly preferable to have a criterion expressed in terms of the given groups A, B, H and such of their subgroups as can be obtained from them by some simple operations. However, the second author conjectured in [4, Conjecture 7.11] the non-existence of such existence criteria, and the present note will confirm this conjecture.

We shall here be concerned with generalized free nilpotent products and generalized free soluble products; they are instances of. *generalized free verbal products*, depending on a verbal subgroup V of the free product of the constituent groups. The precise definition of a generalized free verbal product need not be given here; instead we shall use an existence criterion which — within a suitably narrowed-down class of group amalgams — is necessary and sufficient for the existence of the generalized free verbal product. To the verbal subgroup V there corresponds

a *variety* $\mathfrak{V}$ of groups; thus for example, if V is the second derived group, then $\mathfrak{V}$ consists of the metabelian groups, and the corresponding verbal product is called a metabelian product. The criterion we shall use is the following.

Theorem 1. 1. *If A and B belong to the variety $\mathfrak{V}$ then the generalized free verbal product, corresponding to this variety, of the amalgam*

$$\mathfrak{A} = \mathrm{am}(A, B; H)$$

exists if, and only if, $\mathfrak{A}$ can be embedded in a group of the variety $\mathfrak{V}$.

We omit the proof,[1]) which is not deep but would, of course, require a definition of the generalized free verbal products. The theorem can be extended to amalgams of more than two groups.

It should be noted that this is not a very useful criterion, partly because of the severe restrictions placed upon the amalgam, partly because the embeddability of the amalgam may be very difficult to decide from its intrinsic properties. The use of the theorem lies for us in the substitution of embeddability, a notion with which we are familiar, for existence of a generalized free verbal product, which we have chosen not to define.

The sort of „useful" criterion that one would hope for is formulated in terms of the groups A, B, H defining the amalgam, and groups obtainable from them by the operations of commutation (that is forming from two groups X, Y the commutator group $[X, Y]$), intersection (forming $X \cap Y$), and multiplication (forming XY, under the assumption that this is again a subgroup — for which it suffices that one of X and Y is normal in what they generate). The groups so obtained, together with the trivial group E, form what we call the CIM-*algebra of the amalgam*, and a CIM-criterion will consist of a set of equations between elements of the CIM-algebra.

The CIM-algebra of the amalgam $\mathfrak{A} = \mathrm{am}(A, B; H)$ is in general an incomplete algebra, because the commutator and the product of A and B are not defined in it (except in the trivial case of an „improper" amalgam, when H coincides with A or B).

To illustrate CIM-criteria, we remark that the necessary and sufficient condition for the existence of the generalized direct product of the amalgam $\mathrm{am}(A, B; H)$ is

$$[A, H] = [B, H] = E.$$

This is also sufficient, but not necessary, for the embeddability of the amalgam in a group of the variety $\mathfrak{V}$ provided that both A and B belong to $\mathfrak{V}$. Again it was shown in [4] that the CIM-conditions

$$[A, H, A] = [B, H, B] = \big[A, [B, B] \cap H\big] = \big[B, [A, A] \cap H\big] = E$$

are necessary but not sufficient for the existence of a generalized second nilpotent product of the amalgam.

To prove the non-existence of necessary and sufficient CIM-criteria for the existence of generalized free n-th nilpotent (or n-th soluble) products for $n \geqq 2$, we construct two amalgams of n-th nilpotent (or n-th soluble) groups such that

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[1]) See [4, 4.7].

the CIM-algebras of the two amalgams are isomorphic, and one of the amalgams is, while the other is not, embeddable in an n-th nilpotent (or n-th soluble) group: then both amalgams satisfy precisely the same CIM-conditions, and no set of CIM-conditions can be both necessary and sufficient for embeddability in an n-th nilpotent (or n-th soluble) group; application of Theorem 1. 1 then evidently completes the argument.

In fact by choosing the constituent groups 2nd nilpotent (or metabelian) and by ensuring that the embeddable amalgam is embeddable in a 2nd nilpotent (or metabelian) group, whereas the non-embeddable amalgam is not embeddable in any nilpotent (or soluble) group, whatever the nilpotency class (or solubility length), we deal simultaneously with all values of $n \geqq 2$.

2. Soluble products

We begin with the case of soluble groups because the construction and proof are simpler than for nilpotent groups. We shall arrange our amalgam $\mathfrak{A} = \mathrm{am}(A, B; H)$ so that

$$[A, A] = [A, H] = [B, B] = [B, H] = H \neq E \tag{2. 1}$$

and

$$[H, H] = E.$$

It follows that the CIM-algebra of $\mathfrak{A}$ has only the four elements A, B, H, E, with the composition tables

$[,]$	A	B	H	E
A	H		H	E
B		H	H	E
H	H	H	E	E
E	E	E	E	E

$\cap$	A	B	H	E
A	A	H	H	E
B	H	B	H	E
H	H	H	H	E
E	E	E	E	E

$\cdot$	A	B	H	E
A	A		A	A
B		B	B	B
H	A	B	H	H
E	A	B	H	E

Blank spaces mean „undefined".

It follows from (2. 1) that the groups A and B are metabelian. For our construction we choose them as isomorphic groups of order 56, namely as (splitting) extensions of an elementary abelian group H of order 8 by an automorphism group of order 7. This is not yet sufficient to specify the amalgam, but it is sufficient to ensure the validity of (2. 1), and so the uniqueness (modulo isomorphism) of the CIM-algebras of the amalgams we construct.

Specifically, we put

$$H = \mathrm{gp}(h_1, h_2, h_3; h_i^2 = [h_i, h_j] = 1;\ i, j = 1, 2, 3),$$

and define the automorphisms α, β of H by

$$h_1^\alpha = h_2,\ h_2^\alpha = h_3,\ h_3^\alpha = h_1 h_2,$$

$$h_1^\beta = h_2,\ h_2^\beta = h_3,\ h_3^\beta = h_1 h_3.$$

It is not difficult to verify that both α and β have order 7, and that they jointly generate the group of all automorphisms of H, which is the well-known simple group

of order 168. Now we define groups A, B_1, and B_2 by adjoining to H elements a, b_1, and b_2, respectively, which are to be of order 7 and to induce in H the automorphisms α, α, and β, respectively: thus

$$A = \mathrm{gp}(H, a;\ a^7 = 1,\ h_1^a = h_2,\ h_2^a = h_3,\ h_3^a = h_1 h_2);$$

$$B_1 = \mathrm{gp}(H, b_1;\ b_1^7 = 1,\ h_1^{b_1} = h_2,\ h_2^{b_1} = h_3,\ h_3^{b_1} = h_1 h_2);$$

$$B_2 = \mathrm{gp}(H, b_2;\ b_2^7 = 1,\ h_1^{b_2} = h_2,\ h_2^{b_2} = h_3,\ h_3^{b_2} = h_1 h_3).$$

From these groups we put together two amalgams

$$\mathfrak{A}_1 = \mathrm{am}(A, B_1; H),$$

$$\mathfrak{A}_2 = \mathrm{am}(A, B_2; H).$$

As already remarked, they have the same CIM-algebra. We now show:

Lemma 2. 2. *The amalgam $\mathfrak{A}_1$ is embeddable in a metabelian group, whereas $\mathfrak{A}_2$ is not embeddable in any soluble group.*

PROOF. We denote by C the cyclic group of order 7 generated by an element c, and put

$$G_1 = A \times C.$$

This clearly contains A as a subgroup, and if we put $b_1 = a \times c$, then b_1, like a, induces the automorphism α on H, and $b_1^7 = 1$. The subgroup of G_1 generated by H and b_1 is, therefore, isomorphic to B_1, and if we identify it with B_1, then $A \cap B_1 = H$. Thus G_1 embeds the amalgam $\mathfrak{A}_1$; and G_1 is clearly metabelian.

On the other hand let G_2 be a group in which the amalgam $\mathfrak{A}_2$ is embedded, and assume, as may be done without loss of generality, that $\mathfrak{A}_2$ generates G_2. Now H is normal in A and in B_2, and so H is normal also in G_2. Thus G_2 induces a group of automorphisms in H, and this is isomorphic to $G_2/C(H)$, where $C(H)$ denotes the centralizer of H in G_2; this group of automorphisms contains in particular α and β, the automorphisms induced by a and b_2, and as these generate the group of all automorphisms of H, we see that $G_2/C(H)$ is isomorphic to this, the simple group of order 168. It follows that G_2 is not soluble, and the proof of the lemma is complete.

Combining this lemma with what has been said before, we have thus proved:

Theorem 2. 3. *There is no CIM-criterion that is both necessary and sufficient for the existence of a generalized free soluble product (of arbitrary solubility length $n \geqq 2$) of an amalgam of two groups.*

3. Adjunction of a root to a nilpotent group

The case of nilpotent products requires a more involved construction, and we prepare the ground by exhibiting an example that has arisen in a different context, but will here be put to new use.

The question that the example was designed to answer is this: Let h be an element of a nilpotent group B and let p be a positive integer; is it possible to embed B in a nilpotent group G in which the equation $x^p = h$ has a solution? Under various

restrictions on B the answer was known to be positive, and it was natural to conjecture that it is positive in all cases. This is, however, not true, as the following example will show. Let p be a prime and put

$$B=\mathrm{gp}(b_1, b_2, b_3, \ldots, c_1, c_2, c_3, \ldots, h;\ b_1^p=1,\ b_{i+1}^p=b_i,$$
$$c_1^p=1,\ c_{i+1}^p=c_i,\ [b_i, c_j]=[c_i, h]=1,\ [b_i, h]=c_i \quad (i,j=1, 2, 3, \ldots)).$$

This is a splitting extension of the direct square of the quasi-cyclic group C_{p^∞} by an infinite cyclic group whose generator induces the automorphism that multiplies each element of one C_{p^∞} by the corresponding element of the other (under a fixed isomorphism of the two), and fixes every element of the other C_{p^∞}. Using the nomenclature of MORAN [1], B is the second nilpotent product of an infinite cyclic group and C_{p^∞}.

We now prove that no p-th root of h can be nilpotently adjoined to B, and in fact we prove a little more:[2])

Theorem 3. 1. *Let G be a group that contains B and an element a such that $a^p=h$. Then $a\notin\zeta_\omega(G)$, where*

$$E=\zeta_0(G)\leqq\zeta_1(G)\leqq\ldots\leqq\zeta_\omega(G)\leqq\zeta_{\omega+1}(G)\leqq\ldots$$

is the transfinite upper central series of G; thus in particular $G\neq\zeta_\omega(G)$, and a fortiori G is not nilpotent.

The proof occupies the remainder of this section; we first establish a lemma. The notation is as usual (and some of it has already been used above),

$$x^y=y^{-1}xy=x[x, y],\ [x, y, z]=\big[[x, y], z\big];$$

and

$$G=\gamma_1(G)\geqq\gamma_2(G)\geqq\gamma_3(G)\geqq\ldots$$

is the lower central series of G.

Lemma 3. 2. *Let $m\geqq 2$ be an integer and let K be a group generated by two elements a, b that satisfy (inter alia) the relations*

$$[a^m, b, a^m]=[a^m, b, b]=1.$$

Then for all $i\geqq 3$

(i) *if $g\in\gamma_i(K)$, then $g^{m^2}\in\gamma_{i+1}(K)$, or, differently put, the exponent of $\gamma_i(K)/\gamma_{i+1}(K)$ divides m^2;*

(ii) $$\big[[a, b]^{m^{i-3}}, a^{m^{i-3}}\big]\equiv\big[[a, b]^{m^{i-3}}, b^{m^{i-3}}\big]\equiv 1 \pmod{\gamma_i(K)};$$

(iii) $$[a^{m^{2i-5}}, b]\equiv[a, b^{m^{2i-5}}]\equiv[a, b]^{m^{2i-5}} \pmod{\gamma_i(K)}.$$

PROOF. We use repeatedly the simple remark that

(3. 3) *if $[x, y]\in\gamma_i(K)$ then $[x^r, x^s]\equiv[x, y]^{rs} \pmod{\gamma_{i+1}(K)}$;*

this holds for elements x, y of an arbitrary group K and for arbitrary integers r, s.

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[2]) We owe the knowledge that G is not $\zeta_\omega(G)$ to DR. MICHAEL F. NEWMAN; we had originally only proved that G is not nilpotent, that is to say, $G\neq\zeta_n(G)$ for all finite n.

The proof of the lemma proceeds by induction. As $\gamma_3(K)$ is generated modulo $\gamma_4(K)$ by $[a, b, a]$ and $[a, b, b]$, and as, still modulo $\gamma_4(K)$,

$$[a, b, a]^{m^2} \equiv [[a, b]^m, a^m] \equiv [a^m, b, a^m] \equiv 1,$$

$$[a, b, b]^m \equiv [[a, b]^m, b] \equiv [a^m, b, b] \equiv 1,$$

we see that $\gamma_3(K)/\gamma_4(K)$ has exponent dividing m^2; this proves (i) for $i=3$. Next, trivially,

$$[a, b, a] \equiv [a, b, b] \equiv 1 \pmod{\gamma_3(K)}$$

and

$$[a^m, b] \equiv [a, b^m] \equiv [a, b]^m \pmod{\gamma_3(K)},$$

establishing (ii) and (iii) for $i=3$.

Now assume (i)–(iii) true for $3 \leqq i \leqq n$. If $g \in \gamma_n(K)$ then, by the induction hypothesis, $g^{m^2} \in \gamma_{n+1}(K)$ and if, moreover, k is an arbitrary element of K, then by (3. 3)

$$[g, k]^{m^2} \equiv [g^{m^2}, k] \equiv 1 \pmod{\gamma_{n+2}(K)}.$$

As $\gamma_{n+1}(K)$ is generated modulo $\gamma_{n+2}(K)$ by all such $[g, k]$, with g ranging over $\gamma_n(K)$ and k over K, this establishes (i) for $i=n+1$.

Next, by (ii) for $i=n$,

$$[[a, b]^{m^{n-3}}, a^{m^{n-3}}] \equiv [[a, b]^{m^{n-3}}, b^{m^{n-3}}] \equiv 1 \pmod{\gamma_n(K)};$$

using (i) for $i=n$ and (3. 3), we have in turn

$$[[a, b]^{m^{n-3}}, a^{m^{n-3}}]^{m^2} \equiv [[a, b]^{m^{n-3}}, b^{m^{n-3}}]^{m^2} \equiv 1 \pmod{\gamma_{n+1}(K)},$$

$$[[a, b]^{m^{n-2}}, a^{m^{n-2}}] \equiv [[a, b]^{m^{n-2}}, b^{m^{n-2}}] \equiv 1 \pmod{\gamma_{n+1}(K)},$$

and this proves (ii) for $i=n+1$.

Finally, by (iii) for $i=n$,

$$[a^{m^{2n-5}}, b] \equiv [a, b]^{m^{2n-5}} \pmod{\gamma_n(K)}.$$

We re-write this in the form

$$(a^b)^{m^{2n-5}} = a^{m^{2n-5}}[a, b]^{m^{2n-5}} g,$$

where $g \in \gamma_n(K)$, and thus is central modulo $\gamma_{n+1}(K)$. Also $a^{m^{2n-5}}$ and $[a, b]^{m^{2n-5}}$ commute with each other modulo $\gamma_{n+1}(K)$, by (ii) for $i=n+1$, because $n \geqq 3$ and so $2n-5 \geqq n-2$. Hence

$$(a^b)^{m^{2n-3}} \equiv a^{m^{2n-3}}[a, b]^{m^{2n-3}} g^{m^2} \pmod{\gamma_{n+1}(K)}.$$

But $g^{m^2} \equiv 1 \pmod{\gamma_{n+1}(K)}$ by (i), and we can re-write the congruence in the form

$$[a^{m^{2n-3}}, b] \equiv [a, b]^{m^{2n-3}} \pmod{\gamma_{n+1}(K)},$$

which is the first half of (iii) for $i=n+1$; the other half follows similarly, completing the proof of the lemma.

We turn to the proof of Theorem 3. 1. Assume it false, that is assume $a \in \zeta_\omega(G)$; then there is a finite n such that

$$a \in \zeta_n(G).$$

Put $b=b_{2n-3}$, so that the order of b is p^{2n-3}, and let K be the subgroup of G generated by a and b. Then $a\in\zeta_n(K)$, and K is generated by $\zeta_n(K)$ and b; but $K/\zeta_n(K)$ cannot be a non-trivial cyclic group, hence also $b\in\zeta_n(K)$, and $K=\zeta_n(K)$ is nilpotent of class n. Hence $\gamma_{n+1}(K)=E$. We now apply Lemma 3.2 with $m=p$ and $i=n+1$. Then the congruences are equations, and as $b^{p^{2n-3}}=1$, we have

$$[a^{p^{2n-3}}, b]=1.$$

But $a^p=h$, and so

$$1=[a^{p^{2n-3}}, b]=[h^{p^{2n-4}}, b]=[h, b]^{p^{2n-4}}=c_{2n-3}^{-p^{2n-4}}=c_1^{-1}\neq 1.$$

This contradiction proves the theorem.

We remark that the construction does not use that p was assumed to be a prime, and it can be varied by replacing the C_{p^∞} by a direct product of different such quasi-cyclic groups. Also the theorem is, in a sense, best possible, because one can construct a group G containing B and a p-th root a of h such that $G=\zeta_{\omega+1}(G)$. However, we do not require this fact and omit the proof.

4. Nilpotent products

We shall arrange our amalgam $\mathfrak{A}=\mathrm{am}(A, B; H)$ so that

$$[A, A]=E,\ [B, B]=[B, H]=B'=B'\cap H,\ [B', B]=E, \tag{4.1}$$

and so that A, B, H, B', E are distinct. It follows that the CIM-algebra of $\mathfrak{A}$ has the composition tables

$[,]$	A	B	H	B'	E
A	E		E	E	E
B		B'	B'	E	E
H	E	B'	E	E	E
B'	E	E	E	E	E
E	E	E	E	E	E

$\cap$	A	B	H	B'	E
A	A	H	H	B'	E
B	H	B	H	B'	E
H	H	H	H	B'	E
B'	B'	B'	B'	B'	E
E	E	E	E	E	E

$\cdot$	A	B	H	B'	E
A	A		A	A	A
B		B	B	B	B
H	A	B	H	H	H
B'	A	B	H	B'	B'
E	B	B	H	B'	E

Blank spaces again indicate undefined products.

No CIM-algebra of fewer than five elements would serve our purpose; for in order to make it the CIM-algebra of an amalgam that is embeddable in a 2nd nilpotent group, both A and B must be 2nd nilpotent; but in order to make it the CIM-algebra of another amalgam that is not embeddable in any nilpotent group, we must ensure that not both A and B contain H in the centre, for otherwise the generalized direct product would always provide an embedding in a nilpotent group. It follows that at least one of A and B must be non-abelian, and that the derived group of at least one of A and B must not contain H and so must be distinct from H; and as the amalgam clearly must be proper (that is to say, $A\neq H\neq B$), the CIM-algebra contains 5 elements at least.

It follows from (4. 1) that A is abelian and that B is 2nd nilpotent. For our construction we choose A as a direct product of two infinite cycles and a quasi-cyclic group:

$$A = \mathrm{gp}(a, k, c_1, c_2, c_3, \ldots;\ [a, k] = [a, c_i] = [k, c_i] = 1,$$
$$c_1^p = 1,\ c_{i+1}^p = c_i\ \ (i = 1, 2, 3, \ldots)).$$

For B we take the direct product of an infinite cycle and the group denoted by B in Theorem 3. 1; thus

$$B = \mathrm{gp}(b_1, b_2, b_3, \ldots, c_1, c_2, c_3, \ldots, h, h';\ b_1^p = 1,\ b_{i+1}^p = b_i$$
$$c_1^p = 1,\ c_{i+1}^p = c_i,\ [b_i, c_j] = [b_i, h'] = [c_i, h] = [c_i, h'] =$$
$$= [h, h'] = 1,\ [b_i, h] = c_i\ \ (i, j = 1, 2, 3, \ldots)).$$

We put

$$H = \mathrm{gp}(c_1, c_2, c_3, \ldots, h, h';\ c_1^p = 1,\ c_{i+1}^p = c_i,$$
$$[c_i, h] = [c_i, h'] = [h, h'] = 1\ \ (i = 1, 2, 3, \ldots));$$

the notation indicates how H is embedded as a subgroup of B, and partly how it is embedded in A. To complete the specification, we first identify h with k and h' with a^p, thus making an amalgam $\mathfrak{A}_1$, say; and secondly we identify h with a^p and h' with k, thus making an amalgam $\mathfrak{A}_2$. Both these amalgams satisfy (4. 1); we omit the (easy) verification.

Now $\mathfrak{A}_1$ is embeddable in a 2nd nilpotent group, namely in the generalized direct product of B and the cyclic group generated by a, amalgamating the central element h' of B with a^p. On the other hand, if $\mathfrak{A}_2$ is embedded in a group G, then G is not nilpotent, and even differs from $\zeta_\omega(G)$ by Theorem 3. 1, as $h \in B$ has a p-th root a in A and thus in G. Hence we obtain the theorem announced in the introduction:

Theorem 4. 2. *There is no CIM-criterion that is both necessary and sufficient for the existence of a generalized free nilpotent product (of arbitrary nilpotency class $n \geqq 2$) of an amalgam of two groups.*

References

[1] S. Moran, Associative operations on groups. I, *Proc. London Math. Soc.* (3) **6** (1956), 581 — 596.
[2] B. H. Neumann, An essay on free products of groups with amalgamations, *Phil. Trans. Roy. Soc. London* (A) **246** (1954), 503 — 554.
[3] Otto Schreier, Die Untergruppen der freien Gruppen, *Abh. math. Sem. Hamburg. Univ.* **5** (1927), 161 — 183.
[4] James Wiegold, Nilpotent products of groups with amalgamations, *Publ. Math. Debrecen* **6** (1959), 131 — 168.

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