On the convergence of series of iterates

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Introduction

Let $f(x)$ be an analytic function of the complex argument x. Define the integral iterates $f^{[n]}(x)$ of $f(x)$ recursively by:

$$f^{[0]}(x)=x, \quad f^{[n+1]}(x)=f\{f^{[n]}(x)\} \quad \text{for} \quad n=0, 1, 2, \ldots .$$

Hence $f^{[n+1]}(x)$ is defined at x if and only if $f^{[n]}(x)$ is in the domain of f. It is well known ([1]) that functions $g(x)$ defined by series of the form

$$g(x) = \sum_{n=0}^{\infty} a_n \sum_{r=0}^{n} \binom{n}{r} (-\beta)^{n-r} \Phi\{f^{[r]}(x)\}, \tag{1}$$

for analytic Φ, have applications to functional equations and related fields. If $\beta=1$ and $f(x) = x+1$, then (1) reduces to

$$g(x) = \sum_{n=0}^{\infty} a_n \Delta^n \Phi(x).$$

If $\beta=1$ and $f(x) = \dfrac{x}{x+1}$ while $\Phi(x)=x$, then (1) becomes the factorial series

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!\, a_n}{z(z+1)\ldots(z+n)} \quad \text{where} \quad z = \frac{1}{x}.$$

It was first noted by CAYLEY ([2]) and SCHRÖDER ([3]) that the series

$$\sum_{n=0}^{\infty} \frac{s(s-1)\ldots(s-n+1)}{n!} \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} f^{[r]}(x), \tag{2}$$

when suitably convergent, converge to the generalized iterates $f^{[s]}(x)$, for arbitrary real or complex s. Similarly, by formally differentiating (2) with respect to s and evaluating at $s=0$, one would conjecture ([4]) that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} f^{[r]}(x), \tag{3}$$

when convergent, converges to a function $L(x)$ satisfying the functional differential equation (for a particularly interesting application of this equation, see [5])

$$L\{f(x)\} = L(x)\cdot f'(x). \tag{4}$$

The convergence of the series $\Sigma\, a_n f^{[n]}(x)$, that is the case when $\beta=0$ and $\Phi(x)=x$ in (1), was exhaustively studied by G. JULIA ([6]). The iterative properties of the sum function $f^{[s]}(x)$ of the series (2) were studied by C. BOURLET ([7]), essentially assuming convergence.

Let $\Phi^{[-1]}(x)$ denote the inverse function of the analytic function $\Phi(x)$. By substituting $\Phi^{[-1]}(x)$ into both sides of (1) and replacing $\Phi f \Phi^{[-1]}$ by a new f, one is led to consider the particular case

$$\sum_{n=0}^{\infty} a_n \sum_{r=0}^{n} \binom{n}{r} (-\beta)^{n-r} f^{[r]}(x), \tag{5}$$

since $(\Phi f \Phi^{[-1]})^{[r]} = \Phi f^{[r]} \Phi^{[-1]}$. The present paper develops some necessary and some sufficient conditions for the uniform convergence of (5) when $f(x)$ is analytic about x_0, and $f(x_0)=x_0$. It will be assumed at first that $x_0=0$ since the general case is readily obtained from this case by a simple translation.

Preliminary theorems of iteration theory ([8])

Considerable use will be made of the following well known results from the theory of iteration.

Basic Theorem: *Given a function $f(x)$, analytic about $x=0$ and satisfying $f(0)=0$, $f'(0)=\alpha$ where $0<|\alpha|<1$. Then there exists a $\varrho>0$ and a unique function $F(x)$, the Schröder function for $f(x)$, satisfying:*

(i) $F(x)=\lim_{n\to\infty} \alpha^{-n}\cdot f^{[n]}(x)$, *the sequence converging uniformly in some neighbourhood of $x=0$;*

(ii) *$F(x)$ is analytic about $x=0$, $F(0)=0$ and $F'(0)=1$;*

(iii) *the inverse function, $F^{[-1]}(x)$, exists analytic about $x=0$ and $F^{[-1]}(x) = \sum_{n=1}^{\infty} c_n x^n$, uniformly convergent in $|x|\leqq\varrho$, where $c_1=1$;*

(iv) *$F(x)$ satisfies the Schröder equation $F\{f^{[r]}(x)\}=\alpha^r\cdot F(x)$, whence*

$$f^{[r]}(x) = \sum_{n=1}^{\infty} c_n \alpha^{rn} [F(x)]^n,$$

convergent for $|F(x)|\leqq\varrho$ and all integer $r\geqq 0$;

(v) *the function $\Phi(x;z)$ defined by*

$$\Phi(x;z) = \sum_{n=0}^{\infty} z^{-(n+1)}\cdot f^{[n]}(x)$$

is analytic in the entire complex plane except for simple poles at those $z=\alpha^n$, $n=1,2,\dots$ for which $c_n\neq 0$, and $z=0$ if these poles are not finite in number.

Since the coefficient $c_1 = 1 \neq 0$, the generating function $\Phi(x; z)$ always has a simple pole at $z = \alpha$. Further, since α^n approaches 0 with n when $0 < |\alpha| < 1$, if $\Phi(x; z)$ is analytic at $z = 0$ then $\Phi(x; z)$ must be a rational function of z, having only a finite number of poles in the extended plane. If $\Phi(x; z)$ is not a rational function of z, then $z = 0$ is an essential singularity, a limit point of simple poles.

With the series (5) we associate the series

$$(6) \qquad h(z) = \sum_{n=0}^{\infty} a_n (z-\beta)^n, \text{ convergent say for } |z-\beta| < r.$$

It will be shown that the convergence or divergence of the series (5) is essentially determined by the behaviour of the series (6) at the poles of the generating function $\Phi(x; z)$.

For simplicity, throughout the remainder of this paper the symbols $h(z)$, $\Phi(x; z)$ $F(x)$, c_n, ϱ and r will always have the above meaning relative to $f(x)$, which is assumed to satisfy the hypotheses of the Basic Theorem.

By forming the convex hull of the singularities of $\Phi(x; z)$ it is clear that there will be at least one singularity z_0 of $\Phi(x; z)$ such that the distance from β to z_0 is not exceeded by the distance from β to any other singularity of $\Phi(x; z)$. Hence $z_0 = 0$ or $z_0 = \alpha^k$ for some k such that $c_k \neq 0$, and $|z_0 - \beta| \geqq |\alpha^n - \beta|$ for all n for which $c_n \neq 0$. It has already been shown [9] that if $|z_0 - \beta| < r$, then (5) converges. The peresent paper extends this result, and includes the case $|z_0 - \beta| = r$ and $|z_0 - \beta| > r$.

Principal results

Since the proofs of theorems 1 and 2 below are lengthy. They will be presented after the proof of theorem 4.

Theorem 1. *A sufficient condition that* (5) *converge uniformly in x, for $|F(x)| \leqq \varrho$, is that the series* (6) *converge uniformly on the set of singularities of $\Phi(x; z)$. If $z = 0$ is a limit point of singularities of $\Phi(x; z)$, and if* (6) *converges in $|z - \beta| < |\beta|$, whence $z = 0$ is on the circle of convergence of* (6), *then a sufficient condition for* (5) *to convergence uniformly in $|F(x)| \leqq \varrho |\alpha|^{\nu+1}$ is that*

(i) *for some $\nu > 0$, $N^{-\nu} \left| \sum_{n=1}^{N} a_n \beta^n \right|$ be bounded in N, and that*

(ii) *the singularities of $\Phi(x; z)$, except $z = 0$, lie interior to the intersection of the region $|z - \beta| < |\beta|$ and the angular sector defined by*

$$\arg \beta - \pi/2 + \varepsilon \leqq \arg z \leqq \arg \beta + \pi/2 - \varepsilon$$

for some $\varepsilon > 0$.

PROOF. (postponed).

Theorem 2. *Assume that there exists a K such that $c_K \neq 0$ whence α^K is a simple pole of the generating function, and that for some $0 < \theta < 1$,*

$$\theta |\alpha^K - \beta| \geqq |z - \beta| \text{ for any singularity } z \neq \alpha^K \text{ of } \Phi(x; z).$$

If α^K lies outside the circle of convergence of (6), *then the series* (5) *diverges for all x satisfying $|F(x)| \leqq \varrho$, with the exception of those x for which $F(x) = 0$.*

Proof. (postponed).

Corollary 1. The condition that $n^{(1-\nu)}|a_n\beta^n|$ be bounded in n for some $\nu\neq 0$, or the condition that $\Sigma a_n\beta^n$ be Cesaro summable (C, ν) for some $\nu>0$, implies the condition (i) of theorem 1.

Proof. That summability (C, ν) implies condition (i) is well known [10], as is the first condition since, if bounded by P,

$$\left|N^{-\nu}\sum_{n=1}^{N} a_n\beta^n\right| \leqq N^{-\nu}\sum_{n=1}^{N} n^{(1-\nu)}|a_n\beta^n| n^{\nu-1} \leqq PN^{-\nu}\left\{1+\int_1^N x^{\nu-1}\,dx\right\} =$$

$$= \frac{P}{\nu}\{1+(\nu-1)N^{-\nu}\}, \quad \text{q. e. d.}$$

Theorem 3. *For real α, $0<\alpha<1$, the Cayley—Schröder series* (2) *converges uniformly in $|F(x)|\leqq\varrho\alpha$ if $s\geqq 0$, and uniformly in $|F(x)|\leqq\varrho\alpha^{-s+1}$ if $s<0$.*

Proof. Since a Newton series $\Sigma\binom{s}{n}a_n$ converges in a half line $s>s_0$, it suffices to prove the theorem for $s=-\bar{s}$ where $\bar{s}>0$. In terms of $\bar{s}$ the Cayley—Schröder series becomes

$$\sum_{n=0}^{\infty}(-1)^n\binom{\bar{s}-1+n}{n}\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}f^{[r]}(x),$$

and since $\sum_{n=0}^{\infty}(-1)^n\binom{\bar{s}-1+n}{n}$ is summable [11] $(C,\bar{s})$ for $\bar{s}>0$, it is sufficient by corollary 1 to choose $\nu=\bar{s}$. Finally, since $0<\alpha<1$, it follows that all α^n are in the interval $[0,\alpha]$, and since $\beta=1$, the poles of $\Phi(x;z)$ must lie in the angular sector specified in theorem 1., q. e. d.

Theorem 4. *For real $-1<\alpha<0$, if $f(x)$ satisfies the further property that $f(-x)= -f(x)$, then the continuation* ([9]) *of the Cayley—Schröder series*

$$e^{i\pi s}\sum_{n=0}^{\infty}\binom{s}{n}(-1)^n\sum_{r=0}^{n}\binom{n}{r}(+1)^{n-r}f^{[r]}(x) \tag{7}$$

converges uniformly in $|F(x)|\leqq\varrho|\alpha|$ for $s\geqq 0$, and uniformly in $|F(x)|\leqq\varrho|\alpha|^{-s+1}$ for $s<0$.

Proof. Again it is sufficient to assume $s<0$, so let $s=-\bar{s}$ where $\bar{s}\geqq 0$, and the series in (7) becomes

$$e^{-i\pi\bar{s}}\sum_{n=0}^{\infty}\frac{\Gamma(\bar{s}+n)}{n\Gamma(\bar{s})\Gamma(n)}\sum_{r=0}^{n}\binom{n}{r}(+1)^{n-r}f^{[r]}(x).$$

But since [11]

$$\lim_{n\to\infty}\frac{1}{n^{\bar{s}-1}}\frac{\Gamma(n+\bar{s})}{n\Gamma(n)} = \lim_{n\to\infty}\frac{\Gamma(n+\bar{s})}{n^{\bar{s}}\Gamma(n)} = 1$$

it follows again by corollary 1 that v may be chosen $v=\bar{s}$. It remains to show that the poles of $\Phi(x; z)$ lie in the appropriate angular sector, where in this case $\beta=-1$. But since $f(x)$ is odd, by induction we have

$$f^{[n+1]}(-x)=f^{[n]}\{-f(x)\}=-f^{[n]}\{f(x)\}=-f^{[n+1]}(x),$$

whence all $f^{[n]}(x)$ are odd, and by the Basic Theorem (i) it follows that $F(x)$, and hence $F^{[-1]}(x)$, is also odd. Hence $c_n=0$ when n is even, while $-1<\alpha<0$ implies $-1<\alpha^n<0$ when n is odd, that is, when $c_n\neq 0$. Hence the poles of $\Phi(x; z)$ lie in the proper angular sector, q. e. d.

It should be noted that the series (7) also represents the generalized iterates $f^{[s]}(x)$ since

$$x^s=\{(-1)+(x+1)\}^s=e^{i\pi s}\{1-(x+1)\}^s=e^{i\pi s}\sum_{n=0}^{\infty}\binom{s}{n}(-1)^n(x+1)^n.$$

While the iterative character of these series can be shown directly from the series themselves, as in [7], it will be easier to show these properties after presenting the relations required for the proofs of theorems 1 and 2. At that time the convergence of the series (3), and its relation to the functional differential equation (4) will be discussed.

PROOF OF THEOREMS 1 AND 2:

Lemma 1. *If $f(x)$ satisfies the hypotheses of the basic theorem, then*

$$\sum_{r=0}^{n}\binom{n}{r}(-\beta)^{n-r}f^{[r]}(x)=\sum_{m=1}^{\infty}c_m(\alpha^m-\beta)^n[F(x)]^m, \tag{12}$$

and hence

$$\sum_{n=0}^{N}a_n\sum_{r=0}^{n}\binom{n}{r}(-\beta)^{n-r}f^{[r]}(x)=\sum_{m=1}^{\infty}c_m\left\{\sum_{n=0}^{N}a_n(\alpha^m-\beta)^n\right\}[F(x)]^m, \tag{13}$$

for all $|F(x)|\leqq\varrho$.

PROOF. Both (12) and (13) are finite sums of the expansion of the Basic Theorem (iv) which is convergent for $|F(x)|\leqq\varrho$, q. e. d.

In view of (13), the proof of the first part of theorem 1 is clear, for if the partial sums $\sum_{n=0}^{N}a_n(\alpha^m-\beta)^n$ are uniformly bounded for all N and all m for which $c_m\neq 0$, then (13) can be expected to converge. Similarly if the series (6) diverges at some α^K for which $c_K\neq 0$, then at least one term in the series (13) becomes unbounded and the series can be excepted to diverge. We now proceed with the details.

Specifically set

$$G(M,N,x)=\sum_{m=1}^{M}c_m\left\{\sum_{n=0}^{N}a_n(\alpha^m-\beta)^n\right\}[F(x)]^m \tag{14}$$

or equivalently

$$G(M,N,x)=\sum_{m=1}^{M}c_m(\alpha^m)^{v+1}\left\{\sum_{n=0}^{N}a_n(\alpha^m-\beta)^n\right\}[\alpha^{-(v+1)}F(x)]^m. \tag{14'}$$

If now (6) converges uniformly at all α^m for which $c_m \neq 0$, then the partial sums $\Sigma\, a_n(\alpha^m-\beta)^n$ are uniformly bounded at these α^m, so that there exists a $P>0$ such that

$$\left|\sum_{n=0}^{N} a_n(\alpha^m-\beta)^n\right| \leqq P \text{ for all } N \text{ and all } m, \text{ for which } c_m \neq 0.$$

Applying this to (14), it follows that, for $|F(x)| \leqq \varrho$,

$$\left|\sum_{m=1}^{M} c_m \left\{\sum_{n=0}^{N} a_n(\alpha^m-\beta)^n\right\}[F(x)]^m\right| \leqq \sum_{m=1}^{\infty} |c_m|\,\varrho^m \cdot P,$$

which converges. By the Weiterstrass M test it follows that $\lim\limits_{M\to\infty} G(M, N, x)$ converges uniformly in N and in x, for $|F(x)| \leqq \varrho$. Further, since (6) converges at the α^m for which $c_m \neq 0$, it follows that $\lim\limits_{N\to\infty} G(M, N, x)$ exists for all M, uniformly in $|F(x)| \leqq \varrho$. Hence

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n \sum_{r=0}^{n} \binom{n}{r} (-\beta)^{n-r} f^{[r]}(x) = \lim_{N\to\infty} \lim_{M\to\infty} G(M, N, x) = \lim_{M\to\infty} \lim_{N\to\infty} G(M, N, x) =$$

$$= \lim_{M\to\infty} \sum_{m=1}^{M} c_m h(\alpha^m)[F(x)]^m,$$

which converges uniformly in $|F(x)| \leqq \varrho$ since $h(\alpha^m)$ is bounded when $c_m \neq 0$. This proves the first part of theorem 1. To prove the second part of theorem 1, we need:

Lemma 2. *Given an analytic function $h(z)$ whose expansion $h(z)=\sum\limits_{n=1}^{\infty} a_n(z-\beta)^n$ about $z=\beta$ converges for $|z-\beta|<|\beta|$. If*

$$(15) \qquad N^{-\nu} \sum_{r=1}^{N} a_r \beta^r, \textit{ for some } \nu>0,$$

is bounded for all N, then $z^{\nu+1}h(z)$ and $z^{\nu+1} \sum\limits_{r=1}^{N} a_r(z-\beta)^r$ are uniformly bounded for all N and all z within the intersection of some neighbourhood of $z=0$ and the angular sector

$$(16) \qquad \arg\beta+\pi/2-\varepsilon \geqq \arg z \geqq \arg\beta-\pi/2+\varepsilon$$

for some $\varepsilon>0$.

PROOF. Let P denote an upper bound of (15), then using ABEL's identity

$$\left|z^{\nu+1}\sum_{n=1}^{N} a_n\beta^n\left(\frac{z-\beta}{\beta}\right)^n\right| = |z|^{\nu+1}\left|\sum_{n=1}^{N-1}\left\{\frac{\sum\limits_{r=1}^{n} a_r\beta^r}{n^\nu}\right\} n^\nu\left\{\left(\frac{z-\beta}{\beta}\right)^n-\left(\frac{z-\beta}{\beta}\right)^{n+1}\right\}+\right.$$

$$\left.+N^\nu\left(\frac{z-\beta}{\beta}\right)^N \frac{\sum\limits_{r=1}^{N} a_r\beta^r}{N^\nu}\right| \leqq$$

$$\leqq P\cdot|z|^{\nu+1}\left\{\left|1-\frac{z-\beta}{\beta}\right|\sum_{n=1}^{N-1} n^\nu\left|\frac{z-\beta}{\beta}\right|^n+N^\nu\left|\frac{z-\beta}{\beta}\right|^N\right\},$$

and for z within the circle of convergence $|z-\beta|<|\beta|$, by setting $z=\beta(1-y)$ or equivalently $y=\dfrac{\beta-z}{\beta}$, it follows that $|y|<1$ and that the inequality may be written

$$\leqq 2P|\beta|^{\nu+1}\left\{\frac{|1-y|}{1-|y|}\right\}^{\nu+1}(1-|y|)^{\nu+1}\sum_{n=1}^{\infty} n^{\nu}|y|^{n}. \tag{17}$$

But as a particular case of PRINGSHEIM's theorem (for example see [11], p. 180.) it follows that

$$\lim_{|y|\uparrow 1}(1-|y|)^{\nu+1}\sum_{n=1}^{\infty} n^{\nu}|y|^{n}$$

exists, bounded, and hence since

$$\frac{|1-y|}{1-|y|}$$

is bounded (for example [10], p. 438.) in the angular sector $-\pi/2+\varepsilon\leqq\arg(1-y)\leqq\leqq\pi/2-\varepsilon$, it follows that (17) is uniformly bounded for all y in this sector and in some neighbourhood of $y=1$. But since $z=\beta(1-y)$, y in this region implies z satisfies (16). Since the partial sums are uniformly bounded, clearly $z^{\nu+1}h(z)$ is also bounded, q. e. d.

If, as is assumed in the hypotheses of theorem 1, those α^{n} for which $c_n\neq 0$ lie in the region described in lemma 2, and if (15) holds, then

$$(\alpha_n)^{\nu+1}\sum_{r=1}^{N} a_r(\alpha^{n}-\beta)^{r}$$

is uniformly bounded, say by P_1, for all N and all n for which $c_n\neq 0$. Hence, using (14'), we have that the terms of $G(M, N, x)$ are bounded by

$$\sum_{m=1}^{M}|c_m|P_1|\alpha^{-(\nu+1)}F(x)|^{m}\leqq P_1\sum_{m=1}^{M}|c_m|\varrho^{m}$$

when $|F(x)|\leqq\varrho|\alpha|^{\nu+1}$. Hence using the Weiterstrass M-test, it follows that $\lim_{M\to\infty} G(M, N, x)$ exists, uniformly in N and in x for $|F(x)|\leqq\varrho|\alpha|^{\nu+1}$. Further, for any given M, all the α^{n}, $n\leqq M$, for which $c_n\neq 0$ lie inside the circle of convergence of $h(z)$, so that $\lim_{N\to\infty} G(M, N, x)$ exists for every M, uniformly for $|F(x)|\leqq\varrho|\alpha|^{\nu+1}$. Hence as before

$$\lim_{N\to\infty}\sum_{n=0}^{N} a_n\sum_{r=0}^{n}\binom{n}{r}(-\beta)^{n-r}f^{[r]}(x)=\lim_{N\to\infty}\lim_{M\to\infty} G(M, N, x)=\lim_{M\to\infty}\lim_{N\to\infty} G(M, N, x)=$$

$$=\lim_{M\to\infty}\sum_{m=1}^{M} c_m(\alpha^{m})^{\nu+1}h(\alpha^{m})\cdot[\alpha^{-\nu-1}F(x)]^{m}.$$

But again by lemma 2, $(\alpha^{m})^{\nu+1}h(\alpha^{m})$ is bounded, say by P_2 and for $|F(x)|\leqq\varrho|\alpha|^{\nu+1}$, this last series is majorized by $\sum_{m=1}^{\infty}|c_m|\varrho^{m}P_2$ which converges. This proves theorem 1.

To prove theorem 2, we need the

Lemma 3. (General KOENIG's Theorem). *Let $f(x)$ satisfy the hypothesis of the Basic Theorem. Assume that there exists a K such that $c_K \neq 0$ and that, for some $0 < \theta < 1$,*

$$\theta\,|\alpha^K - \beta| \geqq |\alpha^n - \beta| \quad \textit{whenever} \quad c_n \neq 0.$$

Let

$$g_n(x) = \sum_{s=0}^{n} \binom{n}{s} (-\beta)^{n-s} f^{[s]}(x).$$

Then

$$\lim_{n\to\infty} \frac{g_n(x)}{(\alpha^K - \beta)^n} = c_K [F(x)]^K$$

uniformly in $|F(x)| \leqq \varrho$.

PROOF. It follows from (12) that for $|F(x)| \leqq \varrho$,

$$\left| \frac{g_n(x)}{(\alpha^K - \beta)^n} - c_K [F(x)]^K \right| = \left| \sum_{\substack{m=1 \\ m \neq K}}^{\infty} c_m [F(x)]^m \left\{ \frac{\alpha^m - \beta}{\alpha^K - \beta} \right\}^n \right| \leqq \theta^n \sum_{m=1}^{\infty} |c_m|\, \varrho^n,$$

and since $0 < \theta < 1$, the lemma follows, q. e. d.

In the particular case that $\beta = 0$, then $g_n(x) = f^{[n]}(x)$ and, since $0 < |\alpha| < 1$ it follows that $\theta\,|\alpha| \geqq |\alpha^n|$ for all $n > 1$, for $\theta = |\alpha|$, whence we may choose $K = 1$ and obtain KOENIG's theorem, that is, Basic Theorem (i).

Lemma 4. *Under the hypothesis of lemma 3,*

$$\lim_{n\to\infty} |g_n(x)^{1/n}| = |\alpha^K - \beta|$$

for all x satisfying $|F(x)| \leqq \varrho$ and $F(x) \neq 0$.

PROOF. Since $c_K \neq 0$, and $F(x) \neq 0$ it follows that

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{|g_n(x)|}{|\alpha^K - \beta|^n} = \lim_{n\to\infty} \frac{1}{n} \ln |c_K [F(x)]^K| = 0$$

from which the lemma follows, q. e. d.

We may now prove theorem 2 as follows. Since the radius of convergence of $h(z) = \Sigma\, a_n (z-\beta)^n$ is r, it follows that

$$\overline{\lim_{n\to\infty}} \left| \sqrt[n]{a_n} \right| = \frac{1}{r}.$$

In view of lemma 4, it follows that, for $|F(x)| \leqq \varrho$ and $F(x) \neq 0$,

$$\overline{\lim_{n\to\infty}} |a_n g_n(x)|^{1/n} = \overline{\lim_{n\to\infty}} |a_n|^{1/n} \cdot \lim_{n\to\infty} |g_n(x)|^{1/n} = \frac{|\alpha^K - \beta|}{r}.$$

Hence by the Cauchy root test, if $|\alpha^K - \beta| > r$, the series $\Sigma\, a_n g_n(x)$, that is the series (5), diverges for all x satisfying $|F(x)| \leqq \varrho$ and $F(x) \neq 0$. This completes the proof of theorem 2.

Properties of the series

Theorem 5. *Let $f(x)$ be analytic about $0=f(0)$, and $f'(0)=\alpha$ for real $0<\alpha<1$. Then the series*

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}f^{[r]}(x) \tag{3}$$

converges uniformly in $|F(x)|\leqq \varrho\alpha^{1+\varepsilon}$, for any $\varepsilon>0$, to an analytic function $L(x)$, where

$$L(x)=\ln\alpha\cdot\frac{F(x)}{F'(x)}, \tag{18}$$

and $L(x)$ satisfies the equation

$$L\{f(x)\} = L(x)\cdot f'(x). \tag{19}$$

PROOF. As in theorem 3, the singularities in z of the generating function $\Phi(x;z)$ all lie in the interval $[0,\alpha]$ on the real axis. Since in this case $\beta=1$, and since the series $\Sigma\frac{(-1)}{n}$ converges, the ν of theorem 1 may be chosen $\nu=\varepsilon>0$. Hence (3) converges uniformly in $|F(x)|\leqq\varrho\alpha^{1+\varepsilon}$ to an analytic function $L(x)$. But by (12), each term of this series can be expanded as a power series in $F(x)$ each convergent for $|F(x)|\leqq\varrho$, and hence also for $|F(x)|\leqq\varrho\alpha^{1+\varepsilon}$. Hence by the Weierstrass double series theorem, the order of summation may be interchanged, which yields

$$L(x) = \sum_{m=1}^{\infty}c_m\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\alpha^m-\beta)^n\right\}[F(x)]^m = -\ln\alpha\cdot\sum_{m=1}^{\infty}mc_m[F(x)]^m.$$

But since

$$x=f^{[0]}(x) = \sum_{m=1}^{\infty}c_m[F(x)]^m,$$

differentiation term by term shows that (18) holds. Finally, since $F\{f(x)\}=\alpha F(x)$ while $F'\{f(x)\}\cdot f'(x)=\alpha F'(x)$, clearly (19) follows, q. e. d.

Theorem 6. *If $f(x)$ is analytic about $0=f(0)$, and $f'(0)=\alpha$ for real $0<\alpha<1$, then the functions $f^{[s]}(x)$ defined by*

$$f^{[s]}(x) = \sum_{n=0}^{\infty}\binom{s}{n}\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}f^{[r]}(x), \tag{2}$$

the series being uniformly convergent in $|F(x)|\leqq\varrho\alpha^{-s+1}$ if $s<0$, $|F(x)|\leqq\varrho\alpha$ if $s>0$, satisfy the relations

(i) $$F\{f^{[s]}(x)\}=\alpha^s F(x)$$

(ii) $$f^{[s]}\{f^{[t]}(x)\}=f^{[s+t]}(x),$$

where clearly if s is a positive integer, (2) reduces to the integral iterate of $f(x)$.

PROOF. It is sufficient to prove (i) since $F^{[-1]}(x)$ exists. As in the proof of theorem 5, each term of (2) can be expanded into the power series in $F(x)$ given in

(12), and the order of summation interchanged for x within the region of uniform convergence of (2). But then we obtain

$$f^{[s]}(x) = \sum_{m=1}^{\infty} c_m \left\{ \sum_{n=0}^{\infty} \binom{s}{n} (\alpha^m - 1)^n \right\} [F(x)]^m,$$

and since $0 < \alpha < 1$, this becomes, in view of Basic Theorem (iii),

$$f^{[s]}(x) = \sum_{m=1}^{\infty} c_m [\alpha^s F(x)]^m = F^{[-1]}\{\alpha^s F(x)\}, \quad \text{q. e. d.}$$

Clearly, as in theorem 4, similar theorems hold for $\beta = -1$ and for $f'(0) = \alpha$ where $-1 < \alpha < 0$, provided $f(-x) = -f(x)$.

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