

On functions of three vectors*

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Let X, Y, Z be three vector spaces (not necessarily of the same dimension). We say that a real-valued function f is defined on $X \times Y \times Z$ if, with each ordered triple of vectors x, y, z belonging to X, Y , and Z , respectively, a real number $f(x, y, z)$ is associated.

Theorem. *If a real-valued function on $X \times Y \times Z$ is superadditive in x and y for every z , and subadditive in x and z for every y , then f is additive in x and y for every z , as well as in x and z for every y . In other words, the inequalities*

$$(I) \quad f(x + x_1, y + y_1, z) \geq f(x, y, z) + f(x_1, y_1, z)$$

$$(I') \quad f(x + x_1, y, z + z_1) \leq f(x, y, z) + f(x_1, y, z_1)$$

for all vectors x, x_1, y, y_1, z, z_1 jointly imply the two opposite inequalities.

It goes without saying that f is not necessarily additive in the three vectors x, y, z .

We begin by proving some auxiliary formulae in which $\mathbf{o}$ denotes the vector all components of which are 0. From (I) it follows that $f(\mathbf{o}, \mathbf{o}, z) \geq 2f(\mathbf{o}, \mathbf{o}, z)$ for any z . Hence

$$(1) \quad f(\mathbf{o}, \mathbf{o}, z) \leq 0 \text{ for any } z.$$

Similarly, (I') implies

$$(1') \quad f(\mathbf{o}, y, \mathbf{o}) \geq 0 \text{ for any } y.$$

Hence, in particular,

$$(1^*) \quad f(\mathbf{o}, \mathbf{o}, \mathbf{o}) = 0.$$

Clearly, by virtue of (I) and (I'),

$$(2^*) \quad f(x + x_1, \mathbf{o}, \mathbf{o}) = f(x, \mathbf{o}, \mathbf{o}) + f(x_1, \mathbf{o}, \mathbf{o}),$$

whence

$$(3^*) \quad f(-x, \mathbf{o}, \mathbf{o}) = -f(x, \mathbf{o}, \mathbf{o}) \text{ for any } x.$$

(1*) and (I') imply that $0 = f(\mathbf{o}, \mathbf{o}, \mathbf{o}) \geq f(x, y, \mathbf{o}) + f(-x, -y, \mathbf{o})$. Thus

$$(4) \quad f(x, y, \mathbf{o}) + f(-x, -y, \mathbf{o}) \leq 0 \text{ for any } x \text{ and } y,$$

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and similarly,

$$(4') \quad f(\mathbf{x}, \mathbf{o}, \mathbf{z}) + f(-\mathbf{x}, \mathbf{o}, -\mathbf{z}) \geq 0 \text{ for any } \mathbf{x} \text{ and } \mathbf{z}.$$

From (4) and (4'), for $\mathbf{x} = \mathbf{o}$, in view of (1) and (1'), we obtain

$$(5^*) \quad f(\mathbf{o}, \mathbf{y}, \mathbf{o}) = 0 = f(\mathbf{o}, \mathbf{o}, \mathbf{z}) \text{ for any } \mathbf{y} \text{ and } \mathbf{z}.$$

By (I), (5*), and (I') we find

$$f(\mathbf{o}, -\mathbf{y}, \mathbf{z}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) \leq f(\mathbf{o}, \mathbf{o}, \mathbf{z}) = 0 = f(\mathbf{o}, -\mathbf{y}, \mathbf{o}) \leq f(\mathbf{o}, -\mathbf{y}, \mathbf{z}) + f(\mathbf{o}, -\mathbf{y}, -\mathbf{z})$$

and, consequently,

$$f(\mathbf{o}, \mathbf{y}, \mathbf{z}) \leq f(\mathbf{o}, -\mathbf{y}, -\mathbf{z}) \text{ for any } \mathbf{y} \text{ and } \mathbf{z}.$$

Here, $\mathbf{y}, \mathbf{z}$ and $-\mathbf{y}, -\mathbf{z}$ may change roles, whence

$$(6^*) \quad f(\mathbf{o}, \mathbf{y}, \mathbf{z}) = f(\mathbf{o}, -\mathbf{y}, -\mathbf{z}) \text{ for any } \mathbf{y} \text{ and } \mathbf{z}.$$

By (5*), (I'), (6*), and (I) we have

$$0 = f(\mathbf{o}, \mathbf{y}, \mathbf{o}) \leq f(\mathbf{o}, \mathbf{y}, \mathbf{z}) + f(\mathbf{o}, \mathbf{y}, -\mathbf{z}) = f(\mathbf{o}, \mathbf{y}, \mathbf{z}) + f(\mathbf{o}, -\mathbf{y}, \mathbf{z}) \leq f(\mathbf{o}, \mathbf{o}, \mathbf{z}) = 0.$$

Therefore,

$$(7^*) \quad f(\mathbf{o}, \mathbf{y}, -\mathbf{z}) = -f(\mathbf{o}, \mathbf{y}, \mathbf{z}) = f(\mathbf{o}, -\mathbf{y}, \mathbf{z}) \text{ for any } \mathbf{y} \text{ and } \mathbf{z}.$$

By (I), (4), (5*), and (3*) we find

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}, \mathbf{o}) &\leq -f(-\mathbf{x}, -\mathbf{y}, \mathbf{o}) \leq -f(-\mathbf{x}, \mathbf{o}, \mathbf{o}) - f(\mathbf{o}, -\mathbf{y}, \mathbf{o}) = -f(-\mathbf{x}, \mathbf{o}, \mathbf{o}) = \\ &= f(\mathbf{x}, \mathbf{o}, \mathbf{o}) = f(\mathbf{x}, \mathbf{o}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{o}) \leq f(\mathbf{x}, \mathbf{y}, \mathbf{o}). \end{aligned}$$

Consequently,

$$(8) \quad f(\mathbf{x}, \mathbf{o}, \mathbf{o}) = f(\mathbf{x}, \mathbf{y}, \mathbf{o}) \text{ for any } \mathbf{x} \text{ and } \mathbf{y}.$$

Similarly,

$$(8') \quad f(\mathbf{x}, \mathbf{o}, \mathbf{o}) = f(\mathbf{x}, \mathbf{o}, \mathbf{z}) \text{ for any } \mathbf{x} \text{ and } \mathbf{z}.$$

By (8'), (I), and (I'), we find

$$\begin{aligned} f(\mathbf{x}, \mathbf{o}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) &= f(\mathbf{x}, \mathbf{o}, \mathbf{z}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) \leq f(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq \\ &\leq f(\mathbf{x}, \mathbf{y}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{o}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}). \end{aligned}$$

Therefore

$$(9^*) \quad f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{o}, \mathbf{o}) + f(\mathbf{o}, \mathbf{y}, \mathbf{z}) \text{ for any } \mathbf{x}, \mathbf{y}, \mathbf{z}.$$

From (7*) and (I) it follows that

$$f(\mathbf{x}, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) - f(\mathbf{o}, \mathbf{y}_1, \mathbf{z}) = f(\mathbf{x}, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) + f(\mathbf{o}, -\mathbf{y}_1, \mathbf{z}) \leq f(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

Hence by (9*)

$$f(\mathbf{x}, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) \leq f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{o}, \mathbf{y}_1, \mathbf{z}) = f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}, \mathbf{y}_1, \mathbf{z}) - f(\mathbf{x}, \mathbf{o}, \mathbf{o}).$$

Thus,

$$(10) \quad f(\mathbf{x}, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) \cong f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}, \mathbf{y}_1, \mathbf{z}) - f(\mathbf{x}, \mathbf{o}, \mathbf{o})$$

and, similarly,

$$(10') \quad f(\mathbf{x}, \mathbf{y}, \mathbf{z} + \mathbf{z}_1) \cong f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}, \mathbf{y}, \mathbf{z}_1) - f(\mathbf{x}, \mathbf{o}, \mathbf{o}).$$

From (10), (I'), (8), and (2*) it follows that

$$\begin{aligned} f(\mathbf{x} + \mathbf{x}_1, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) &\cong f(\mathbf{x} + \mathbf{x}_1, \mathbf{y}, \mathbf{z}) + f(\mathbf{x} + \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}) - f(\mathbf{x} + \mathbf{x}_1, \mathbf{o}, \mathbf{o}) \cong \\ &\cong f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}_1, \mathbf{y}, \mathbf{o}) + f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}) + f(\mathbf{x}, \mathbf{y}_1, \mathbf{o}) - f(\mathbf{x} + \mathbf{x}_1, \mathbf{o}, \mathbf{o}) = \\ &= f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}_1, \mathbf{o}, \mathbf{o}) + f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}) + f(\mathbf{x}, \mathbf{o}, \mathbf{o}) - f(\mathbf{x} + \mathbf{x}_1, \mathbf{o}, \mathbf{o}) = \\ &= f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}). \end{aligned}$$

Therefore

$$(11) \quad f(\mathbf{x} + \mathbf{x}_1, \mathbf{y} + \mathbf{y}_1, \mathbf{z}) \cong f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z})$$

and, similarly,

$$(11') \quad f(\mathbf{x} + \mathbf{x}_1, \mathbf{y}, \mathbf{z} + \mathbf{z}_1) \cong f(\mathbf{x}, \mathbf{y}, \mathbf{z}) + f(\mathbf{x}_1, \mathbf{y}, \mathbf{z}_1).$$

This completes the proof of our theorem.

In conclusion, we mention a corollary of (8) and (8'). If the vector space $\mathbf{Z}$ has the dimension 0, then f is independent of $\mathbf{y}$. If the dimension of $\mathbf{Y}$ is 0, then f is independent of $\mathbf{z}$.

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