On Ogawa's Criterion For Univalence*

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1. Introduction

In a recent note, OGAWA obtained the following result ([2], p. 8).

Theorem 1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic for* $|z| \leqq 1$, *with* $f'(z) \neq 0$ *there, let* $\varphi(t)$ *denote a single-valued real function, sufficiently smooth for all t, and let k denote a real constant. If the inequality*

$$\int_{\theta_1}^{\theta_2} d\left[\arg\left[e^{i\theta} f'(e^{i\theta})\right] + k\varphi\left(f(e^{i\theta})\right)\right] > -\pi, \quad z = re^{i\theta}, \tag{1}$$

holds for all $\theta_1 < \theta_2$, *then* $f(z)$ *is univalent for* $|z| < 1$.

OGAWA then applied Theorem 1 to the special cases (i) $\varphi(f(z)) = \Im(f(z))$ and (ii) $\varphi(f(z)) = \arg f(z)$ to obtain various sufficient conditions for the univalence (or p-valence) of analytic functions $f(z)$ [3].

In this note we obtain certain representations of functions $f(z)$ satisfying an inequality of the form (1) for the two elementary cases alluded to above. These representations yield some results which are modest extensions of OGAWA's original results; they also raise other questions, answers to which should prove interesting.

2. Principal Results

Most of the results of this note depend upon the following lemma which is a simple extension of one due to KAPLAN ([1], p. 174).

Lemma. *Let* $p(\theta)$ *denote a real-valued function that is of bounded variation for* $0 \leqq \theta \leqq 2\pi$, *and let* $p(\theta)$ *satisfy*

$$\int_{0}^{2\pi} dp(\theta) = 2\pi(k+1), \tag{2}$$

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where k is a real constant, and

$$\int_{\theta_1}^{\theta_2} dp(\theta) > -\pi, \tag{3}$$

for all $\theta_1 < \theta_2$. Then there exists a real-valued non-decreasing function $q(\theta)$ such that

$$\int_0^{2\pi} dq(\theta) = 2\pi(k+1) \tag{4}$$

and

$$|p(\theta) - q(\theta)| \leq \frac{\pi}{2} \tag{5}$$

hold. Moreover, if $P(r, \theta)$ and $Q(r, \theta)$ denote the functions

$$\begin{aligned} P(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \mathscr{R}\left(\frac{1+ze^{-i\alpha}}{1-ze^{-i\alpha}}\right)[p(\alpha)-(k+1)\alpha]\,d\alpha, \quad z = re^{i\theta}, \\ Q(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \mathscr{R}\left(\frac{1+ze^{-i\alpha}}{1-ze^{-i\alpha}}\right)[q(\alpha)-(k+1)\alpha]\,d\alpha, \quad z = re^{i\theta}, \end{aligned} \tag{6}$$

harmonic for $r < 1$, and if $F(z)$ and $G(z)$ are analytic (not necessarily single-valued) functions satisfying

$$\begin{aligned} P(r,\theta) &= \arg F(z), \quad |F(0)| = 1, \quad \arg 1 = 0, \\ Q(r,\theta) &= \arg G(z), \quad |G(0)| = 1, \quad \arg 1 = 0, \end{aligned} \tag{7}$$

then

$$\mathscr{R}\left(\frac{F(z)}{G(z)}\right) \geq 0 \tag{8}$$

and

$$\frac{\partial}{\partial\theta} \arg[z^{k+1}G(z)] \geq 0 \tag{9}$$

hold for $0 < |z| \leq 1$.

Proof. A proof of this lemma is contained, for all practical purposes, in Kaplan's note ([1], p. 174), hence we shall omit the details. However, we do note that the condition (9) is the usual one that guarantees that $w = z^{k+1}G(z)$ maps each circle $|z| = r$ onto a curve that is (spiral and) star-like with respect to $w = 0$; if k is a nonnegative integer, then the image of each circle $|z| = r$ under $w = z^{k+1}G(z)$ is a closed (and star-like) curve.

We now apply our lemma.

Theorem 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic for $|z| \leq 1$ with $f'(z) \neq 0$ and $f(z)/z \neq 0$ there, and let m denote a real constant. Then $f(z)$ satisfies the inequality*

$$\int_{\theta_1}^{\theta_2} \mathscr{R}\left[1 + e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})} + me^{i\theta}f'(e^{i\theta})\right] d\theta > -\pi \tag{10}$$

for all $\theta_1 < \theta_2$ *if and only if there exists a close-to-convex function* $H(z) = z + \sum_{n=2}^{\infty} b_n z^n$, *univalent for* $|z| < 1$, *such that*

$$H(z) = \frac{e^{mf(z)} - 1}{m}, \qquad mH(z) + 1 \neq 0. \tag{11}$$

PROOF. We first suppose that there exists some real m such that $f(z)$ satisfies (10) for all $\theta_1 < \theta_2$. We apply the Lemma, with $p(\theta) = \arg\,[e^{i\theta} f'(e^{i\theta}) e^{mf(e^{i\theta})}]$, with $k = 0$, to obtain the functions $F(z)$ and $G(z)$ satisfying (8) and (9). But here it is clear that $F(z) = f'(z) e^{mf(z)}$, so that (9) becomes

$$\mathscr{R}\left(\frac{f'(z)\,e^{mf(z)}}{G(z)}\right) \geqq 0. \tag{12}$$

Since $G(z)$ satisfies (9) with $k = 0$, we conclude that $zG(z)$ is a starlike and univalent function for $|z| < 1$. Hence there exists a convex and univalent function $\Phi(z)$, with $\Phi(0) = 0$, $|\Phi'(0)| = 1$ such that

$$\mathscr{R}\left(\frac{f'(z)\,e^{mf(z)}}{\Phi'(z)}\right) \geqq 0$$

holds for $|z| \leqq 1$; this implies that $e^{mf(z)}$ is univalent and close-to-convex for $|z| < 1$ ([1], p. 169). One part of the theorem now follows.

The remaining part of the theorem is now a matter of verifying the inequality for a function of the form (11).

Remark 1. If for some real m, $f(z)$ satisfies (10), for all $\theta_1 < \theta_2$, it follows that $H(z)$ in (11) is univalent. Therefore $f(z)$, is univalent too. This is one of OGAWA's results ([2], p. 9).

Remark 2. If $f(z)$ satisfies (10), then the m cannot be arbitrary. For, since $H(z)$ is a univalent function that does not take on the value $-\frac{1}{m}$, it follows from the $\frac{1}{4}$ theorem that $|m| \leqq 4$. Indeed, if $H(z)$ is univalent, and if m is chosen so that $mH(z) + 1 \neq 0$, for $|z| < 1$, then $f(z)$ defined by (11) is univalent too; one uses the close-to-convex property of $H(z)$ in order to establish (10).

Theorem 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic for* $|z| \leqq 1$ *with* $f'(z) \neq 0$ *and* $f(z)/z \neq 0$ *there, and let* m *denote a real constant. Then*

$$\mathscr{R}\left[1 + z\frac{f''(z)}{f'(z)} + mzf'(z)\right] \geqq 0 \tag{13}$$

holds for $|z| \leqq 1$ *if and only if* $f(z)$ *is univalent for* $|z| < 1$ *and has the representation*

$$f(z) = \frac{1}{m}\log\left(1 + m\Phi(z)\right), \quad 1 + m\Phi(z) \neq 0, \tag{14}$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} c_n z^n$ *is a (normalized) univalent convex function.*

PROOF. Suppose that (13) holds for $|z| \leqq 1$. Then the standard procedure relative to positive harmonic functions yields the HERGLOTZ representation

$$1 + z\frac{f''(z)}{f'(z)} + mzf'(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-i\alpha}}{1-ze^{-i\alpha}}\, d\mu(\alpha) \tag{15}$$

where $\mu(\alpha)$ is non-decreasing and satisfies

$$\int_0^{2\pi} d\mu = 2\pi. \tag{16}$$

From (15) and (16) we obtain, after several simple operations,

$$e^{mf(z)} - 1 = m\Phi(z), \tag{17}$$

where

$$\Phi(z) = \int_0^z e^{-\frac{1}{\pi}\int_0^{2\pi} \log(1-ze^{-i\alpha})d\mu(\alpha)}\, dz. \tag{18}$$

It is well-known that $\Phi(z)$ in (18) is a (normalized) convex function univalent for $|z| < 1$. It now follows that $f(z)$ is also univalent.

The converse can be readily verified by several simple computations.

Remark 3. OGAWA showed that (13) implies $f(z)$ is not only univalent, but also convex in the direction of the imaginary axis ([2], p. 9). This last property of $f(z)$ follows easily from the representation (17); all one must do is study the variation of $\arg f(z)$.

Theorem 4. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic for* $|z| \leqq 1$, *with* $f'(z) \neq 0$ *and* $f(z)/z \neq 0$ *there, and let* m *denote a real constant. Then a necessary and sufficient condition that*

$$\int_{\theta_1}^{\theta_2} \mathscr{R}\left[1 + e^{i\theta}\frac{f''(e^{i\theta})}{f'(e^{i\theta})} + me^{i\theta}\frac{f'(e^{i\theta})}{f(e^{i\theta})}\right] d\theta > -\pi \tag{19}$$

hold for all $\theta_1 < \theta_2$ *is that there exist a function* $G(z)$, *analytic for* $|z| < 1$, *with* $G(0) = 0$ *and* $|G'(0)| = 1$, *for which* (9) *(with* $m = k$*) and*

$$\mathscr{R}\left(\frac{f'(z)[f(z)]^m}{z^m G(z)}\right) \geqq 0 \tag{20}$$

hold.

PROOF. Suppose that (19) holds for all $\theta_1 < \theta_2$, for some real m. Then we apply our lemma, with $p(\theta) = \arg\{f'(e^{i\theta})[f(e^{i\theta})/e^{i\theta}]^m\}$, $m = k$, to obtain the proper $G(z)$ and $F(z) = f'(z)\left[\frac{f(z)}{z}\right]^m$ such that (9), (8) — and hence (20) — hold.

As for the other part of the theorem, it is a simple matter to verify the result. Indeed, this last is one of OGAWA's particular results ([3], pp. 432—436).

Remark 4. OGAWA managed to show that (19) implies that $f(z)$ is univalent; this we have not been able to do by the methods of the present note.

Remark 5. Again, the constant m is not arbitrary; it must at least satisfy the inequality $m > -\frac{1}{2}$.

3. Concluding Remarks

The classes of functions introduced by OGAWA have special properties that should be investigated; distortion inequalities, coefficient inequalities geometric properties, etc. It would also be of interest to determine the complete range of permissible values for the constant m that intervenes. It would be of interest to determine the radius of star-likeness, the radius of convexity, and the radius of "close-to-convexity" for the members of the various classes discussed in this note. Finally, it would be of some interest to determine whether or not each function $f(z)$ that is convex in the direction of the imaginary axis has a representation analogous to (17), and thus obtain some of UMEZAWA's results [4].

References

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