

On transformations of certain hypergeometric functions of three variables

By H. M. SRIVASTAVA (Jodhpur)

1. Introduction

In the year 1893 LAURICELLA ([5]) conjectured the existence of ten hypergeometric functions of three variables in addition to F_A, F_B, F_C and F_D defined and studied by him. These ten functions, namely $F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S$ and F_T , have been defined and studied by SARAN ([7]). In §§ 3 and 4 of this paper I have obtained certain hypergeometric transformations associated with the functions F_F, F_P and F_R .

It may be remarked here that the functions F_P and F_R chosen by me have not been dealt with hitherto¹⁾, so far as their transformations are concerned. On the other hand, no transformation into a known function has been previously obtained for F_F . PANDEY [6], however, gave the transformation

$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ = (1-x-z)^{-\beta_1}(1-y)^{-\beta_2} G_A \left(1-\gamma_1, \beta_1, \beta_2; \gamma_2; \frac{x}{x+z-1}, \frac{1}{y-1}, \frac{z}{x+z-1} \right),$$

where $\gamma_1 + \gamma_2 = 1 + \alpha_1$, and the new function

$$G_A(\alpha, \beta, \beta'; \gamma; x, y, z) = \\ = \sum_{m, n, p=0}^{\infty} \frac{(\alpha, n+p-m)(\beta, m+p)(\beta', n)}{(1, m)(1, n)(1, p)(\gamma, n+p-m)} x^m y^n z^p.$$

The four linear transformations of F_F , given by me in this paper, connect this function with either F_M or F_N . Besides, I have been able to show that, under certain conditions, the function F_F is reducible to Appell's F_4 .

In the last section I have given some interesting cases of reducibility of the remaining functions $F_E, F_G, F_K, F_M, F_N, F_S$ and F_T . Some of these reductions yield very interesting results by putting two of the variables equal. For instance, by putting $y = z$, the reduction formulae for F_S lead to well-known relations between Appell's and Gauss's hypergeometric functions.

¹⁾ Vide [8], p. 294, last paragraph.

2. Definitions

Following the notation of [7] the ten hypergeometric functions of three variables are defined by the equalities

$$\begin{aligned}
 (2.1) \quad & F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n+p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} x^m y^n z^p \\
 & \quad [r + (\sqrt{s} + \sqrt{t})^2 = 1];
 \end{aligned}$$

$$\begin{aligned}
 (2.2) \quad & F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p \\
 & \quad [(1-s)(s-t) = rs];
 \end{aligned}$$

$$\begin{aligned}
 (2.3) \quad & F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p \\
 & \quad [r+s=1=r+t];
 \end{aligned}$$

$$\begin{aligned}
 (2.4) \quad & F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n)(\gamma_3, p)} x^m y^n z^p \\
 & \quad [(1-r)(1-s) = t];
 \end{aligned}$$

$$\begin{aligned}
 (2.5) \quad & F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p \\
 & \quad [r+t=1=s];
 \end{aligned}$$

$$\begin{aligned}
 (2.6) \quad & F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n)(\alpha_3, p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p \\
 & \quad [(1-r)s + (1-s)t = 0];
 \end{aligned}$$

$$\begin{aligned}
 (2.7) \quad & F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+p)(\alpha_2, n)(\beta_1, m+n)(\beta_2, p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p
 \end{aligned}$$

$$\begin{aligned}
 & [(st - s - t)^2 = 4rst]; \\
 (2.8) \quad & F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m+p)(\alpha_2, n)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)} x^m y^n z^p \\
 & [s(1 - \sqrt{r})^2 + (1-s)t = 0];
 \end{aligned}$$

$$\begin{aligned}
 (2.9) \quad & F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)} x^m y^n z^p \\
 & [r+s = rs, \quad s = t];
 \end{aligned}$$

and

$$\begin{aligned}
 (2.10) \quad & F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \\
 & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)} x^m y^n z^p \\
 & [r - rs + s = t];
 \end{aligned}$$

where, as usual, $(\lambda, k) = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$, and $|x| < r$, $|y| < s$ and $|z| < t$.

3. Transformations of F_F .

BAILEY ([3]) has proved the theorem that

$$(3.1) \quad F_4[\alpha, \beta; \gamma, \gamma'; z(1-Z), Z(1-z)] = {}_2F_1(\alpha, \beta; \gamma; z) {}_2F_1(\alpha, \beta; \gamma'; Z),$$

provided that $\gamma + \gamma' = \alpha + \beta + 1$, this formula being valid inside simply connected regions surrounding $z=0, Z=0$ for which

$$|z(1-Z)|^{1/2} + |Z(1-z)|^{1/2} < 1.$$

From definition we have

$$\begin{aligned}
 & F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\
 & = \sum_{n=0}^{\infty} \frac{(\alpha_1, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} F_4(\alpha_1 + n, \beta_1; \gamma_1, \gamma_2 + n; x, z) y^n.
 \end{aligned}$$

Therefore, using (3.1) we get

$$\begin{aligned}
 (3.2) \quad & F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1 + \alpha_1 + \beta_1 - \gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] = \\
 & = \sum_{n=0}^{\infty} \frac{(\alpha_1, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} {}_2F_1(\alpha_1 + n, \beta_1; 1 + \alpha_1 + \beta_1 - \gamma_2; x) \cdot {}_2F_1(\alpha_1 + n, \beta_1; \gamma_2 + n; z) y^n.
 \end{aligned}$$

Applying Euler's transformation [2]

$$(3.3) \quad {}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} {}_2F_1\left(\gamma-\alpha, \beta; \gamma; \frac{z}{z-1}\right),$$

we obtain

$$(3.4) \quad \begin{aligned} & F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] = \\ & = (1-z)^{-\beta_1} \sum_{n=0}^{\infty} \frac{(\alpha_1, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} {}_2F_1(\alpha_1+n, \beta_1; 1+\alpha_1+\beta_1-\gamma_2; x) \cdot \\ & \quad \cdot {}_2F_1\left(\gamma_2-\alpha_1, \beta_1; \gamma_2+n; \frac{z}{z-1}\right) y^n. \end{aligned}$$

On expanding Gauss's functions ${}_2F_1$ inside the summation and comparing the resulting triple series with (2.6), we obtain the transformation

$$(3.5) \quad \begin{aligned} & F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] = \\ & = (1-z)^{-\beta_1} F_N\left(\beta_1, \gamma_2-\alpha_1, \beta_2, \alpha_1, \beta_1, \alpha_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x, \frac{z}{z-1}, y\right). \end{aligned}$$

Now, using the transformation [2]

$$(3.6) \quad {}_2F_1(\alpha, \beta; \gamma; x) = (1-x)^{-\alpha} {}_2F_1\left(\alpha, \gamma-\beta; \gamma; \frac{x}{x-1}\right),$$

in (3.4) and comparing the resulting triple series with (2.6), we get the transformation

$$(3.7) \quad \begin{aligned} & F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] = \\ & = (1-x)^{-\alpha_1} (1-z)^{-\beta_1} F_N\left(1+\alpha_1-\gamma_2, \gamma_2-\alpha_1, \beta_2, \alpha_1, \beta_1, \alpha_1; \right. \\ & \quad \left. 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; \frac{x}{x-1}, \frac{z}{z-1}, \frac{-y}{x-1}\right). \end{aligned}$$

If we use the relation [2]

$$(3.8) \quad {}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; z)$$

in (3.2) and compare the resulting triple series with (2.5), we obtain the transformation

$$(3.9) \quad \begin{aligned} & F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1+\alpha_1+\beta_1-\gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] = \\ & = (1-z)^{\gamma_2-\alpha_1-\beta_1} F_M(\beta_1, \beta_2, \beta_2, \alpha_1, \gamma_2-\alpha_1, \alpha_1; \gamma_1, \gamma_2, \gamma_2; x, z, y), \end{aligned}$$

and if we use both (3. 6) and (3. 8) in (3. 2) and proceed as above, the transformation

$$(3. 10) \quad \begin{aligned} & F_F[\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; 1 + \alpha_1 + \beta_1 - \gamma_2, \gamma_2, \gamma_2; x(1-z), y, z(1-x)] = \\ & = (1-x)^{-\alpha_1} (1-z)^{\gamma_2 - \alpha_1 - \beta_1} F_M \left(1 + \alpha_1 - \gamma_2, \beta_2, \beta_2, \alpha_1, \gamma_2 - \alpha_1, \alpha_1; \right. \\ & \quad \left. \gamma_1, \gamma_2, \gamma_2; \frac{x}{x-1}, z, \frac{-y}{x-1} \right) \end{aligned}$$

is obtained, both the transformations being valid when $\gamma_2 = \beta_1 + \beta_2$ and $\gamma_1 + \gamma_2 = 1 + \alpha_1 + \beta_1$.

Now, let us write

$$(3. 11) \quad \begin{aligned} & F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ & = \sum_{p=0}^{\infty} \frac{(\alpha_1, p)(\beta_1, p)}{(1, p)(\gamma_2, p)} F_2(\alpha_1 + p, \beta_1 + p, \beta_2; \gamma_1, \gamma_2 + p; x, y) z^p. \end{aligned}$$

Using then the transformation [1]

$$(3. 12) \quad F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = (1-y)^{-\alpha} F_2 \left(\alpha, \beta, \gamma' - \beta'; \gamma, \gamma' \frac{x}{1-y}, \frac{y}{y-1} \right)$$

and expanding the resulting Appell's function F_2 inside the summation, we get the interesting reduction

$$(3. 13) \quad \begin{aligned} & F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \gamma_2 - \beta_1, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, -y) = \\ & = (1-y)^{-\alpha_1} F_4 \left(\alpha_1, \beta_1; \gamma_1, \gamma_2; \frac{-x}{y-1}, \frac{y}{y-1} \right). \end{aligned}$$

4. Transformations of F_P and F_R

On writing

$$(4. 1) \quad \begin{aligned} & F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ & = \sum_{p=0}^{\infty} \frac{(\alpha_1, p)(\beta_2, p)}{(1, p)(\gamma_2, p)} F_2(\beta_1, \alpha_1 + p, \alpha_2; \gamma_1, \gamma_2 + p; x, y) z^p, \end{aligned}$$

and using the relation (3. 12), we obtain

$$(4. 2) \quad \begin{aligned} & F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \gamma_2 - \alpha_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ & = (1-y)^{-\beta_1} H_A \left(\alpha_1, \beta_1, \gamma_2 - \alpha_2; \gamma_1, \gamma_2; \frac{x}{1-y}, \frac{-y}{1-y}, z \right), \end{aligned}$$

where the new function

$$(4. 3) \quad \begin{aligned} & H_A(\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z) = \\ & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\alpha, m+p)(\beta, m+n)(\beta', n+p)}{(1, m)(1, n)(1, p)(\gamma, m)(\gamma', n+p)} x^m y^n z^p. \end{aligned}$$

Similarly, on rewriting

$$(4.4) \quad \begin{aligned} & F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ & = \sum_{n=0}^{\infty} \frac{(\alpha_2, n)(\beta_1, n)}{(1, n)(\gamma_2, n)} F_2(\alpha_1, \beta_1 + n, \beta_2; \gamma_1, \gamma_2 + n; x, z) y^n, \end{aligned}$$

and using (3.12), we get the transformation

$$(4.5) \quad \begin{aligned} & F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \gamma_2 - \alpha_2; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ & = (1-z)^{-\alpha_1} H_A \left(\alpha_1, \beta_1, \alpha_2; \gamma_1, \gamma_2; \frac{x}{1-z}, y, \frac{-z}{1-z} \right). \end{aligned}$$

The equality (2.8) can be written as

$$(4.6) \quad \begin{aligned} & F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ & = \sum_{n=0}^{\infty} \frac{(\alpha_2, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} F_4(\alpha_1, \beta_1; \gamma_1, \gamma_2 + n; x, z) y^n. \end{aligned}$$

Using the relation [1]

$$(4.7) \quad \begin{aligned} & F_4 \left(\alpha, \alpha + \frac{1}{2}; \gamma, \frac{1}{2}; x, y \right) = \frac{1}{2} (1 + \sqrt{y})^{-2\alpha} {}_2F_1 \left[\alpha, \alpha + \frac{1}{2}; \gamma; \frac{x}{(1 + \sqrt{y})^2} \right] + \\ & + \frac{1}{2} (1 - \sqrt{y})^{-2\alpha} {}_2F_1 \left[\alpha, \alpha + \frac{1}{2}; \gamma; \frac{x}{(1 - \sqrt{y})^2} \right], \end{aligned}$$

we obtain

$$(4.8) \quad \begin{aligned} & F_R \left(\alpha_1, \alpha_2, \alpha_1, \alpha_1 + \frac{1}{2}, \beta_2, \alpha_1 + \frac{1}{2}; \frac{1}{2}, \gamma_2, \gamma_2; x, y, z \right) = \\ & = \frac{1}{2} (1 + \sqrt{x})^{-2\alpha_1} F_3 \left[\alpha_2, \alpha_1, \beta_2, \alpha_1 + \frac{1}{2}; \gamma_2; y, \frac{z}{(1 + \sqrt{x})^2} \right] + \\ & + \frac{1}{2} (1 - \sqrt{x})^{-2\alpha_1} F_3 \left[\alpha_2, \alpha_1, \beta_2, \alpha_1 + \frac{1}{2}; \gamma_2; y, \frac{z}{(1 - \sqrt{x})^2} \right]. \end{aligned}$$

5. Some reducible cases

On writing

$$(5.1) \quad \begin{aligned} & F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \\ & = \sum_{p=0}^{\infty} \frac{(\alpha_1, p)(\beta_2, p)}{(1, p)(\gamma_3, p)} F_2(\alpha_1 + p, \beta_1, \beta_2 + p; \gamma_1, \gamma_2; x, y) z^p, \end{aligned}$$

and using the relation [1]

$$(5.2) \quad F_2(\alpha, \beta, \beta'; \beta, \gamma'; x, y) = (1-x)^{-\alpha} {}_2F_1\left(\alpha, \beta'; \gamma'; \frac{y}{1-x}\right),$$

we obtain

$$(5.3) \quad \begin{aligned} F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \beta_1, \gamma_2, \gamma_3; x, y, z) = \\ = (1-x)^{-\alpha_1} F_4\left(\alpha_1, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x}, \frac{z}{1-x}\right). \end{aligned}$$

Similarly, we get

$$(5.4) \quad \begin{aligned} F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \beta_1, \gamma_2, \gamma_2; x, y, z) = \\ = (1-x)^{-\alpha_1} F_1\left(\alpha_1; \beta_2, \beta_3; \gamma_2, \frac{y}{1-x}, \frac{z}{1-x}\right), \end{aligned}$$

$$(5.5) \quad \begin{aligned} F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \beta_2, \gamma_3; x, y, z) = \\ = (1-y)^{-\alpha_2} F_2\left(\beta_1, \alpha_1, \alpha_2; \gamma_1, \gamma_3; x, \frac{z}{1-y}\right), \end{aligned}$$

$$(5.6) \quad \begin{aligned} F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1, \gamma_2, \gamma_3; x, y, z) = \\ = (1-x)^{-\beta_1} F_2\left(\alpha_2, \beta_2, \beta_1; \gamma_2, \gamma_3; y, \frac{z}{1-x}\right), \end{aligned}$$

$$(5.7) \quad \begin{aligned} F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1, \gamma_2, \gamma_2; x, y, z) = \\ = (1-x)^{-\beta_1} F_1\left(\alpha_2, \beta_2, \beta_1; \gamma_2; y, \frac{z}{1-x}\right) \end{aligned}$$

and

$$(5.8) \quad \begin{aligned} F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \alpha_1, \gamma_2, \gamma_2; x, y, z) = \\ = (1-x)^{-\beta_1} F_3\left(\alpha_2, \alpha_3, \beta_2, \beta_1; \gamma_2; y, \frac{z}{1-x}\right). \end{aligned}$$

It may be mentioned that the transformation (5.8) was previously given by PANDEY [6] by using Pochhammer's double loop type integral for F_N , namely

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \varrho + \varrho', \varrho + \varrho'; x, y, z) = \frac{\Gamma(1-\varrho)\Gamma(1-\varrho')\Gamma(\varrho + \varrho')}{(2\pi i)^2}.$$

$$\cdot \int (-t)^{\varrho-1} (t-1)^{\varrho'-1} {}_2F_1(\alpha_2, \beta_2; \varrho; ty) F_2[\beta_1, \alpha_1, \alpha_3; \gamma_1, \varrho'; x, (1-t)z] dt,$$

where $|x| + |(1-t)z| < 1$ and $|ty| < 1$ along the contour.

Also, on writing

$$(5.9) \quad \begin{aligned} F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \\ = \sum_{n=0}^{\infty} \frac{(\alpha_2, n)(\beta_2, n)}{(1, n)(\gamma_2, n)} F_2(\beta_1, \alpha_1, \alpha_2 + n; \gamma_1, \gamma_2 + n; x, z) y^n, \end{aligned}$$

and using the transformation [1]

$$(5.10) \quad F_2(\alpha, \beta, \beta'; \gamma, \beta'; x, y) = (1-y)^{-\alpha} {}_2F_1\left(\alpha, \beta; \gamma; \frac{x}{1-y}\right),$$

we obtain the interesting reduction

$$(5.11) \quad \begin{aligned} F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \alpha_2, \alpha_2; x, y, z) = \\ = (1-y)^{-\beta_2} (1-z)^{-\beta_1} {}_2F_1\left(\alpha_1, \beta_1; \gamma_1; \frac{x}{1-z}\right). \end{aligned}$$

Now, writing

$$(5.12) \quad \begin{aligned} F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \\ = \sum_{m=0}^{\infty} \frac{(\alpha_1, m)(\beta_1, m)}{(1, m)(\gamma_1, m)} F_1(\alpha_2, \beta_2, \beta_3; \gamma_1 + m; y, z) x^m, \end{aligned}$$

and using the transformation [1]

$$(5.13) \quad \begin{aligned} F_1(\alpha, \beta, \beta'; \gamma; x, y) = \\ = (1-x)^{\gamma-\alpha-\beta} (1-y)^{-\beta'} F_1\left(\gamma-\alpha, \gamma-\beta-\beta', \beta'; \gamma; x, \frac{y-x}{y-1}\right), \end{aligned}$$

we get

$$(5.14) \quad \begin{aligned} F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \\ = (1-y)^{\alpha_1-\beta_2} (1-z)^{-\beta_3} F_1\left(\alpha_1, \beta_1, \beta_3; \gamma_1; x+y-xy, \frac{z-y}{z-1}\right), \end{aligned}$$

where $\gamma_1 = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 + \beta_3$.

Two particular cases of (5.14) are

$$(5.15) \quad \begin{aligned} F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; -y, y, z) = \\ = (1-y)^{\alpha_1-\beta_2} (1-z)^{-\beta_3} F_1\left(\alpha_1, \beta_1, \beta_3; \gamma_1; y^2, \frac{z-y}{z-1}\right) \end{aligned}$$

and

$$(5.16) \quad \begin{aligned} F_S\left(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; \frac{y}{1-y}, y, z\right) = \\ = (1-y)^{\alpha_1-\beta_2}(1-z)^{-\beta_3} F_1\left(\alpha_1, \beta_1, \beta_3; \gamma_1; 2y, \frac{z-y}{z-1}\right), \end{aligned}$$

where, as before, $\gamma_1 = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 + \beta_3$.

It is interesting to note that by putting $y=z$ in (5.14), we get the well-known transformations [1]

$$\begin{aligned} F_3(\alpha_1, \gamma_1 - \alpha_1, \beta_1, \gamma_1 - \beta_1; \gamma_1; x, y) = \\ = (1-y)^{\alpha_1+\beta_1-\gamma_1} {}_2F_1[\alpha_1, \beta_1; \gamma_1; 1 - (1-x)(1-y)] \end{aligned}$$

and

$$\begin{aligned} F_1\left(\alpha_1, \beta_1, \beta_2 + \beta_3; \beta_1 + \beta_2 + \beta_3; x, \frac{y}{y-1}\right) = \\ = (1-y)^{\alpha_1} {}_2F_1[\alpha_1, \beta_1; \beta_1 + \beta_2 + \beta_3; x + (1-x)y], \end{aligned}$$

the latter being a particular case of (5.18) below.

Lastly, writing

$$(5.17) \quad \begin{aligned} F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \\ = \sum_{m=0}^{\infty} \frac{(\alpha_1, m)(\beta_1, m)}{(1, m)(\gamma_1, m)} F_1(\alpha_2, \beta_2, \beta_1 + m; \gamma_1 + m; y, z) x^m = \\ = \sum_{n=0}^{\infty} \frac{(\alpha_2, n)(\beta_2, n)}{(1, n)(\gamma_1, n)} F_1(\beta_1, \alpha_1, \alpha_2 + n; \gamma_1 + n; x, z) y^n, \end{aligned}$$

and using the relation [1]

$$(5.18) \quad F_1(\alpha, \beta, \beta'; \beta + \beta'; x, y) = (1-y)^{-\alpha} {}_2F_1\left(\alpha, \beta; \beta + \beta'; \frac{x-y}{1-y}\right),$$

we get the reductions

$$(5.19) \quad \begin{aligned} F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \gamma_1 - \beta_1, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \\ = (1-y)^{-\alpha_2} F_1\left(\beta_1, \alpha_1, \alpha_2; \gamma_1; x, \frac{z-y}{1-y}\right) = \\ = (1-z)^{-\alpha_2} F_3\left(\alpha_1, \alpha_2, \beta_1, \gamma_1 - \beta_1; \gamma_1; x, \frac{y-z}{1-z}\right); \end{aligned}$$

and

$$\begin{aligned}
 (5.20) \quad F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2; x, y, z) = \\
 = (1-x)^{-\beta_1} F_1\left(\alpha_2, \beta_2, \beta_1; \alpha_1 + \alpha_2; y, \frac{z-x}{1-z}\right) = \\
 = (1-z)^{-\beta_1} F_3\left(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1 + \alpha_2; \frac{x-z}{1-z} y\right).
 \end{aligned}$$

On putting $y=z$ in (5. 19) and $x=z$ in (5. 20), we obtain the interesting reductions

$$(5.21) \quad F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, y) = (1-y)^{-\alpha_2} {}_2F_1(\alpha_1, \beta_1; \gamma_1; x),$$

provided that $\gamma_1 = \beta_1 + \beta_2$, and

$$(5.22) \quad F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_1, \gamma_1; x, y, x) = (1-x)^{-\beta_1} {}_2F_1(\alpha_2, \beta_2; \gamma_1; y),$$

valid when $\gamma_1 = \alpha_1 + \alpha_2$.

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