A new proof of a theorem of Delange

By ALFRÉD RÉNYI (Budapest)

§ 1. Introduction

A complex-valued function $g(n)$ defined on the set of natural numbers is called a *multiplicative number theoretical function* if

$$g(n\cdot m)=g(n)\cdot g(m) \quad \text{for} \quad (n, m)=1. \tag{1. 1}$$

Here and in what follows (n, m) denotes the greatest common divisor of n and m; if besides (1. 1) one has for any prime p

$$g(p^\alpha)=g(p) \quad \text{for} \quad \alpha=2, 3, \ldots \tag{1. 2}$$

then $g(n)$ is called *strongly multiplicative.*

A complex-valued function $f(n)$ defined on the set of natural numbers is called an *additive number theoretical function* if

$$f(n\cdot m) = f(n)+f(m) \quad \text{for} \quad (n, m)=1 \tag{1. 3}$$

and *strongly additive* if besides (1. 3) for every prime p

$$f(p^\alpha)=f(p) \quad \text{for} \quad \alpha=2, 3, \ldots. \tag{1. 4}$$

The theorem of DELANGE in question is the following (see [1] and [2]):

Theorem of Delange: Let $g(n)$ be a strongly multiplicative (complex-valued) number theoretical function such that

$$|g(n)|\leqq 1 \quad \textit{for} \quad n=1, 2, \ldots \tag{1. 5}$$

Suppose that the series

$$\sum_p \frac{g(p)-1}{p} \tag{1. 6}$$

is convergent, where p runs over all primes). Then the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(n) = M(g) \tag{1. 7}$$

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*) Throughout this paper p always denotes a prime.

exists, and one has

$$M(g) = \prod_p \left(1 + \frac{g(p)-1}{p}\right). \tag{1.8}$$

Remark. Delange has proved also that the convergence of the series (1. 6) is also necessary for the existence of a *non-zero* mean value $M(g)$ of g, under the condition (1. 5), but in what follows we shall deal only with the sufficiency of the convergence of the series (1. 6) for the existence of (1. 7).

H. DELANGE ([1], [2]) has given two different proofs for this theorem. The first is analytic and makes use of deep Tauberian theorems; the second is elementary but rather difficult, it is based on Selberg's sieve method. In what follows we give a third very simple and elementary proof.

This proof is based on the inequality: *If $f(n)$ is a strongly additive number theoretical function,*

$$A(N) = \sum_{p \leqq N} \frac{f(p)}{p} \tag{1.9}$$

and

$$B^2(N) = \sum_{p \leqq N} \frac{|f(p)|^2}{p} \tag{1.10}$$

then

$$\frac{1}{N} \sum_{n=1}^{N} |f(n) - A(N)|^2 \leqq C_1 B^2(N) \tag{1.11}$$

where C_1 is a positive constant, not depending on the function $f(n)$.

We shall call (1. 11) the *Turán—Kubilius inequality*, because the inequality (1. 10) was proved by J. KUBILIUS ([3], p. 35, Lemma 3. 1.)) by generalizing an earlier inequality of P. TURÁN ([4]).

Let us note that the Turán—Kubilius inequality has an obvious probabilistic interpretation. Let us consider the finite probability space P_N consisting of the first N positive integers, these being supposed to be equiprobable. A strongly additive function $f(n)$ can be considered as a random variable on P_N and can be represented in the form

$$f(n) = \sum_{p \leqq N} \varepsilon_p(n) f(p) \qquad (n = 1, 2, \ldots, N) \tag{1.12}$$

where

$$\varepsilon_p(n) = \begin{cases} 1 & \text{if } p \mid n \\ 0 & \text{if } p \nmid n. \end{cases} \tag{1.13}$$

(Here and in what follows $p \mid n$ denotes that p is a divisor of n and $p \nmid n$ that p does not divide n.) Clearly *)

$$\frac{1}{N} \sum_{m=1}^{N} f(m) = \sum_{p \leqq N} f(p) \frac{1}{N} \left[\frac{N}{p}\right] = \sum_{p \leqq N} \frac{f(p)}{p} + O\left(\frac{B(N)}{\sqrt{\log N}}\right).$$

*) $[x]$ denotes the integral part of x.

Thus (1. 11) can be written also in the equivalent form

$$\frac{1}{N}\sum_{n=1}^{N}\left|f(n)-\frac{1}{N}\sum_{m=1}^{N}f(m)\right|^2 \leqq C_2 B^2(N) \tag{1. 11'}$$

where C_2 is an absolute constant. As clearly the variance of the random variable $\varepsilon_p(n)$ on P_N is equal to $\frac{1}{p}\left(1-\frac{1}{p}\right)+O\left(\frac{1}{N}\right)$, if the random variables $\varepsilon_p(n)$ $(p \leqq N)$ were pairwise independent, then the variance of the sum $f(n)=\sum_{p \leqq N}\varepsilon_p(n)f(p)$ would be exactly equal to the sum of the variances of the summands; now while the variables $\varepsilon_p(n)$ are not exactly independent, their dependence is rather weak, and this is reflected in (1. 11') which says (for real valued f) that the variance (on P_N) of the sum $f(n)=\sum_{p \leqq N}\varepsilon_p(n)f(p)$ is less than an absolute constant multiplied by the sum of the variances of the terms of this sum.

The importance of the theorem of Delange is shown by the remark, that it implies one of the main theorems of the statistical theory of additive number theoretic functions, namely the theorem of P. ERDŐS and A. WINTNER [5]. Recently in [6] we gave a new proof of this theorem by the standard methods of analytic number theory. When writing the paper [6] we were unaware of the fact that there exists such a simple proof of the theorem of Delange as presented below. After having obtained such a proof, the simplest way to obtain the Erdős—Wintner theorem is that through the theorem of Delange. This way of obtaining the Erdős—Wintner theorem is much simpler than that given in [6]. Let us mention that recently a quite different approach to the Erdős—Wintner theorem has been given by E. NOVOSELOV ([7]).

§ 2. Proof of Delange's theorem by the Turán—Kubilius inequality

Let us mention first that it is very easy to see (see [1]) that if $g_1(n)$ and $g_2(n)$ are strongly multiplicative functions, such that $|g_1(n)| \leqq 1$ and $|g_2(n)| \leqq 1$ $(n=1, 2, \ldots)$ and if

$$g_1(p)=g_2(p)$$

except for $p \in P$ where P is a set of primes such that $\sum_{p \in P}\frac{1}{p} < +\infty$, then if $M(g_2)$ exists, so does $M(g_1)$ and if the formula (1. 8) holds for $g=g_2$ it holds for $g=g_1$ too. As further it follows from the convergence of the series (1.6) and the condition (1. 5) that the series *)

$$\sum_{p}\frac{1-\Re(g(p))}{p} \tag{2. 1}$$

*) $\Re(z)$ denotes the real part of the complex number z.

is convergent too and its terms are nonnegative, and consequently the series

$$\sum_{\Re(g(p))\leqq\frac{1}{2}} \frac{1}{p} \tag{2. 2}$$

is convergent too, we may suppose that

$$\Re(g(p))>\frac{1}{2} \tag{2. 3}$$

for all primes p, because if there would be primes p for which $\Re(g(p))\leqq\frac{1}{2}$, we could, in view of the convergence of the series (2. 2) change the value of $g(p)$ for these primes (to 1 say) so that (2. 3) should be satisfied for all primes p. From now on we shall therefore suppose that (2. 3) holds for all primes p. Let us put

$$|g(p)|=r(p) \quad \text{and} \quad \arg g(p)=\vartheta(p) \tag{2. 4}$$

where $-\pi<\vartheta(p)\leqq+\pi$, i. e. we suppose that $g(p)=r(p)e^{i\vartheta(p)}$. It follows easily from the convergence of the series (1. 6) and the condition (2. 3) that the series with positive terms

$$\sum_p \frac{\log\dfrac{1}{r(p)}}{p} \tag{2. 5}$$

$$\sum_p \frac{\log^2 r(p)}{p} \tag{2. 6}$$

$$\sum_p \frac{\vartheta^2(p)}{p} \tag{2. 7}$$

are convergent and the series

$$\sum_p \frac{\vartheta(p)}{p} \tag{2. 8}$$

is convergent (however the series (2. 8) is in general not absolutely convergent).

Let us define the strongly multiplicative function $g_N(n)$ by putting

$$g_N(p)=\begin{cases} g(p) & \text{if} \quad p\leqq\log N \\ 1 & \text{if} \quad p>\log N. \end{cases} \tag{2. 9}$$

By other words we put for any natural number n

$$g_N(n)=\prod_{\substack{p/n \\ p\leqq\log N}} g(p) \tag{2. 10}$$

Let us put further

$$h_N(n)=\sum_{d/n}\mu\left(\frac{n}{d}\right)g_N(d) \tag{2. 11}$$

where d runs over all (positive) divisors of n and $\mu(n)$ is the Möbius function. Then $h_N(n)$ is also a multiplicative function, (but not strongly multiplicative, because $h_N(p)=g_N(p)-1$ and $h_N(p^\alpha)=0$ for $\alpha\geqq 2$).

It is easy to see that

$$g_N(n) = \sum_{d|n} h_N(d) \tag{2.12}$$

and the series $\sum_{n=1}^{\infty} \frac{h_N(n)}{n}$ is convergent and we have

$$\sum_{n=1}^{\infty} \frac{h_N(n)}{n} = \prod_{p \leqq \log N} \left(1 + \frac{g(p)-1}{p}\right). \tag{2.13}$$

We have clearly in view of (2.12)

$$\frac{1}{N} \sum_{n=1}^{N} g_N(n) = \frac{1}{N} \sum_{d=1}^{N} h_N(d) \left[\frac{N}{d}\right] = \sum_{d=1}^{\infty} \frac{h_N(d)}{d} + R_N \tag{2.14}$$

where*)

$$|R_N| \leqq \frac{1}{N} \sum_{d=1}^{\infty} |h_N(d)| \leqq \frac{1}{N} \prod_{p \leqq \log N} (1 + |g(p)-1|) \leqq \frac{2^{\pi(\log N)}}{N} \tag{2.15}$$

and thus

$$\lim_{N \to \infty} R_N = 0 \tag{2.16}$$

As further

$$\lim_{N \to \infty} \prod_{p \leqq \log N} \left(1 + \frac{g(p)-1}{p}\right) = \prod_{p} \left(1 + \frac{g(p)-1}{p}\right) \tag{2.17}$$

the convergence of the infinite product on the right of (2.17) (extended over all primes p) being a consequence of the convergence of the series (1.6), we have proved that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g_N(n) = \prod_{p} \left(1 + \frac{g(p)-1}{p}\right). \tag{2.18}$$

Thus in order to prove (1.7) it is sufficient to show that putting

$$D_N = \frac{1}{N} \sum_{n=1}^{N} (g(n) - g_N(n)) \tag{2.19}$$

we have

$$\lim_{N \to \infty} D_N = 0. \tag{2.20}$$

Now we clearly have

$$D_N = \frac{1}{N} \sum_{n=1}^{N} g_N(n) \Big(\prod_{\substack{p|n \\ p > \log N}} g(p) - 1 \Big) \tag{2.21}$$

and thus, putting

$$f_N(n) = \sum_{\substack{p|n \\ p > \log N}} \log g(p) \tag{2.22}$$

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*) $\pi(x)$ denotes as usual the number of primes $p \leqq x$.

we have

$$D_N \leqq \frac{1}{N}\sum_{n=1}^{N} |e^{f_N(n)}-1| \tag{2.23}$$

Now we need the simple inequality

$$|e^z-1| \leqq |z|$$

valid for $\Re(z)\leqq 0$, which can be proved e. g. as follows

$$|e^z-1| = \left|\int_0^z e^\zeta d\zeta\right| \leqq \int_0^{|z|} dx = |z|.$$

It follows, in view of $\Re\big(f_N(n)\big)\leqq 0$

$$D_N \leqq \frac{1}{N}\sum_{n=1}^{N} |f_N(n)|. \tag{2.24}$$

By Cauchy's inequality we get

$$D_N^2 \leqq \frac{1}{N}\sum_{n=1}^{N} |f_N(n)|^2. \tag{2.25}$$

Now evidently $f_N(n)$ is a strongly additive function, and thus, in view of the inequality $|a+b|^2 \leqq 2\big(|a|^2+|b|^2\big)$

$$\frac{1}{N}\sum_{n=1}^{N} |f_N(n)|^2 \leqq \frac{2}{N}\sum_{n=1}^{N}\left|f_N(n)-\sum_{p\leqq N}\frac{f_N(p)}{p}\right|^2 + 2\left|\sum_{p\leqq N}\frac{f_N(p)}{p}\right|^2. \tag{2.26}$$

As regards the second term on the right of (2. 26) we have

$$\left|\sum_{p\leqq N}\frac{f_N(p)}{p}\right|^2 = \left|\sum_{\log N<p\leqq N}\frac{\log r(p)}{p}\right|^2 + \left|\sum_{\log N<p\leqq N}\frac{\vartheta(p)}{p}\right|^2$$

and therefore in view of the convergence of the series (2. 5) and (2. 8) we have

$$\lim_{N\to\infty}\left|\sum_{p\leqq N}\frac{f_N(p)}{p}\right|^2 = 0. \tag{2.27}$$

On the other hand, we can apply to the first term on the right of (2. 25) the Turán—Kubilius inequality, and obtain

$$\frac{1}{N}\sum_{n=1}^{N}\left|f_N(n)-\sum_{p\leqq N}\frac{f_N(p)}{p}\right|^2 \leqq C_1\cdot\sum_{p\leqq N}\frac{|f_N(p)|^2}{p} = C_1\left(\sum_{\log N<p\leqq N}\frac{\log^2 r(p)+\vartheta^2(p)}{p}\right).$$

In view of the convergence of the series (2. 6) and (2. 7) it follows that

$$\lim_{N\to\infty}\frac{2}{N}\sum_{n=1}^{N}\left|f_N(n)-\sum_{p\leqq N}\frac{f_N(p)}{p}\right|^2 = 0. \tag{2.28}$$

Thus we obtain from (2. 25), (2. 26), (2. 27) and (2. 28) that (2. 20) holds. In view of what has been said above, this completes the proof of the theorem of Delange

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