Applications of analytic number theory to the study of type sets of torsionfree Abelian groups II.

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We continue our study begun in [3] of the problem proposed by BEAUMONT and PIERCE [1] of finding necessary and sufficient conditions that a set of types be the type set of a torsionfree Abelian group. The principal results here are an improved sufficient condition for the rank two case, a necessary and sufficient condition for a type set to be the type set of a quotient divisible rank two group, generalizing [3] and KOEHLER [5], and a construction, generalizing those in [2], [3], and CORNER [7], of great utility in type considerations; for example, every rank two group has a type set that is the type set of some $A(S; x)$ of rank two, and an $A(S; x)$ of finite rank is quotient-divisible if and only if $\tau(A(S; x))$ is non-nil, where $\tau(A(S; x))$ is the g.l.b. of the types of a maximal independent subset. In the last paragraph we formulate the rank two problem in number-theoretic terms which clearly show its analytic character.

For most terms see FUCHS [4], § 42, or BEAUMONT and PIERCE [1]; and also see [3]. Let $S: \tau_1, \tau_2, \ldots$ be a sequence of types and T a set of types. S is a τ_0-sequence, where τ_0 is a type, if for all $i<j$, $\tau_i \cap \tau_j = \tau_0$; T is a τ_0-set if $\tau' \neq \tau''$ implies $\tau' \cap \tau'' = \tau_0$, for all τ', τ'' in T. The formula $D(i, j, p)$ stands for the predicate $\alpha_0(p) < \alpha_i(p) < \alpha_j(p) = \infty$, where α_n are characteristics. Let α_n be a characteristic belonging to τ_n, $n=0,1, \ldots$, and suppose S is a τ_0-sequence, $T=\{\tau_n; n=1, 2, \ldots\}$. Then T is a τ_0-set. We say that τ_i is a *snarl of the subsequence* S': $\tau_{n(1)}, \tau_{n(2)}, \ldots$ (here $n(1) < n(2) < \ldots$) if $D(i, n(j), p(j))$ holds for an infinite set of primes $p(j)$. The subsequence S' is *free* if no τ_i is a snarl of it (briefly, if it has no snarls). Snarl of a subset and free subset are defined analogously. By the *standard list* we mean a well-ordered set comprising all coprime pairs (a, b) of integers with $a \geqq 0$, such that if $\max\{a, |b|\} < \max\{a', |b'|\}$ then (a, b) precedes (a', b') in the order. A characteristic is *non-nil* if it is almost everywhere zero or infinity. A torsionfree group is *quotient-divisible* (q. d.) if it contains a free subgroup whose quotient is a divisible torsiongroup. The sequence S, above, is a *type sequence* of the rank two group G if for some independent x and y in G, the standard list can be indexed so that the type of $a_n x + b_n y$ is τ_n. The following Lemmas from [3] are used extensively:

Lemma. *Let G be a rank two group containing independent elements x and y. Define α_0 by: $\alpha_0(p) = \min\{H_p(x), H_p(y)\}$ for every p. Let (a, b) and (c, d) be coprime pairs, let p be a prime and suppose $H_p(ax+by) > \alpha_0(p)$. Then:* 1. *$T(G)$ is an $[\alpha_0]$-set*

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and every type sequence of G is an $[\alpha_0]$-sequence. 2. *If x' and y' have equal type greater than $[\alpha_0]$ then x' and y' are dependent.* 3. *The determinant $ad-bc$ is divisible by p if and only if $H_p(cx+dy)>\alpha_0(p)$.* 4. *If $H_p(ax+by)=\infty>\alpha_0(p)$ then $H_p(cx+dy)=\alpha_0(p)+H_p(ad-bc)$ (the last height is computed in the additive group Z of all integers).*

We now define the group $A(S;x)$. Let S be a linearly independent set of reals in the open interval (0, 1), and let x be a function whose domain is S and whose range is a subset of the cartesian product $\prod_p I(p)$, where $I(p)$ is the group of all p-adic integers and p runs over all primes. For every s in S, $x(s)$ is a function on the set of all primes whose value at p, denoted by $x(s,p)$, is a p-adic integer. Let $A(S;x)$ be the set of all finite rational combinations $\sum a_s s$ such that for every p, $\sum a_s x(s,p)$ is a p-adic integer. Then $A(S;x)$ is a torsionfree group of rank $|S|$. The height at p of a member $\sum a_s s$ of $A(S;x)$ is the height at p of $\sum a_s x(s,p)$ in $I(p)$ which is simply the p-adic value of $\sum a_s x(s,p)$. For every p, the correspondence $\sum a_s s \to \sum a_s x(s,p)$ is a homomorphism which preserves heights at p; its kernel is the set of all members of $A(S;x)$ of infinite height at p and its image is p-pure in $I(p)$. For a subset S' of S let $S'(p)=\{(s,x(s,p));\, s\in S'\}$. We say that $S'(p)$ is *independent* provided that a (finite) rational combination $\sum a_s x(s,p)$ is zero only if all a_s are zero. Then $S'(p)$ is independent if and only if every nonzero member $\sum_{s\in S'} a_s s$ of $A(S;x)$ has finite height at p. Thus, if $S(p)$ is independent then $A(S;x)$ is isomorphic with a p-pure subgroup of $I(p)$. Let K be a set of primes and suppose that for every p in K and every s in S, $H_p(x(s,p))=\alpha(p)<\infty$ and that the zero degree term in the standard p-adic power series representation of $p^{-\alpha(p)}x(s,p)$ is $[|\log p|^s]$. Then for every nonzero $a=\sum a_s s$ in $A(S;x)$, $H_p(a)=\alpha(p)$ for almost all p in K; cf. [2]. To see this, suppose that K' is an infinite subset of K and that for every p in K', $H_p(a)\geqq 1+\alpha(p)$. (It is clear that $H_p(a)\geqq\alpha(p)$ for all p in K.) Then for all large p in K',

$$p>\sum|a_s|\,|\log p|\geqq|\sum a_s[|\log p|^s]|,$$

while the latter quantity is divisible by p, hence zero; but this same quantity tends to infinity with p along with its (nonzero) term of maximum exponent. Thus we have a contradiction. We remark, finally, that $A(S;x)$ is cohesive if and only if for every p, either $S(p)$ is independent or else $x(s,p)=0$ for all s (see Theorem 1(*b*) of [2]); if we require that x be a one-to-one function and that $S(p)$ be algebraically independent over the rationals, in the sense that the *set* of all $x(s,p)$, for s in S, is algebraically independent, then we can construct rigid systems of cohesive groups $A(S;x)$ as in Theorems 4 and 5, and Corollary, of [2].

The expression $[|\log p|^s]$ appears so often that we shall abbreviate it by $L(p,s)$. As we saw above, if p divides $\sum a_s L(p,s)$ for every p in a set F_1, and if not all a_s are zero, then F_1 is finite.

Theorem 1. *Let T be a set of types closed under finite g. l. b. (if τ and τ' belong to T then so does $\tau\cap\tau'$). Then there exists a group $A(S;x)$ whose type set is T and whose rank is $|T|$.*

PROOF. Let S be a linearly independent set of real numbers in (0, 1) of the same power as T. Let each type in T be represented by a characteristic of that type and

let α be a one-to-one function mapping S onto the resulting set of characteristics, with values $\alpha(s)$, for s in S. The value of $\alpha(s)$ at p will be denoted by $\alpha(s, p)$. Let x be a function from S into $\prod_p I(p)$ satisfying (*a*) for every p the relation $S_p = \{(s, x(s, p)); \alpha(s, p) < \infty\}$ is independent, i.e. if $\sum a_s x(s, p) = 0$ then all a_s are zero; (*b*) for every p and every s in S, $H_p(x(s, p)) = \alpha(s, p)$, and if $\alpha(s, p) < \infty$ then $x(s, p) = p^{\alpha(s, p)}(L(s, p) + \ldots)$. To get such an x, take, for each p, a linearly independent subset of elements $y(s, p)$ in $I(p)$ having $L(s, p)$ as zero degree term, and in one to one correspondence $s \leftrightarrow y(s, p)$ with S. Then let $x(s, p) = p^{\alpha(s, p)} y(s, p)$, if $\alpha(s, p) < \infty$, $x(s, p) = 0$ if $\alpha(s, p) = \infty$. Now $A(S; x)$ will have the required properties provided: for every nonzero $a = \sum a_s s$ in $A(S; x)$, the type of a is the g.l.b. of the types of its nonzero summands $a_s s$ (with $a_s \neq 0$). We show this by induction on the number of summands. For the case of only one summand we have $H_p(s) = H_p(x(s, p)) = \alpha(s, p)$ for every p so that the type of $a = a_s s$ is the type of s. Now suppose the claim valid for all elements of $A(S; x)$ with fewer than n summands, where $n > 1$, and let a have exactly n summands. For every p set

$$\alpha_0(p) = \min \{H_p(s); a_s \neq 0\} = \min \{\alpha(s, p); a_s \neq 0\} = \min \{H_p(x(s, p)); a_s \neq 0\}.$$

By condition (*a*) and the remarks preceding the statement of Theorem 1, the summands of a with finite height add up to an element of finite height, at p, while those of infinite height add up to an element of infinite height, at p. This shows that $H_p(a) = \infty$ if and only if $\alpha_0(p) = \infty$. Let F be the set of all primes p such that if $a_s \neq 0$ then $H_p(a_s s) = H_p(s)$ and every sum of fewer than n summands of a has height at p equal to the g.l.b. of *its* summands. By the induction hypothesis, F has finite complement. The induction is now completed by showing that

$$F_1 = \{p; p \in F \quad \text{and} \quad H_p(a) > \alpha_0(p)\}$$

is finite. The induction hypothesis implies that if p belongs to F_1 and $a_s \neq 0$ then $H_p(a_s) < \infty$, and therefore that $H_p(a) < \infty$. Let a_t be nonzero and let p belong to F_1. Then $H_p(a_t t) = H_p(t) = \alpha_0(p)$; for supposing that $H_p(a_t t) > \alpha_0(p)$ we deduce from the induction hypothesis that $\sum_{s \neq t} a_s s$, having fewer than n summands, has height at p equal to $\alpha_0(p)$ (recall that p belongs to $F_1 \subset F$), which implies that

$$H_p(a) = H_p(a_t t + \sum_{s \neq t} a_s s) = H_p(\sum_{s \neq t} a_s s) = \alpha_0(p),$$

which contradicts the membership of p in F_1. We now see that if p belongs to F_1 and $a_s \neq 0$ then $H_p(a) \geqq 1 + \alpha_0(p) = 1 + \alpha(s, p) < \infty$. By condition (*b*), p divides $\sum a_s L(p, s)$ for every p in F_1, so F_1 is finite. The proof is complete.

From now on we consider mainly rank two groups, with a slight change in notation. The set S will be a two element set, $S = \{x, y\}$, where x and y are independent reals in (0, 1). The functions are denoted by u and v, and $A(x, y; u, v)$ is the group of all rational combinations $ax + by$ for which for all p, $au_p + bv_p$ is a p-adic integer; where u_p, for example, is the value of u at p, i.e., a p-adic integer. It may be that u and v are the same.

Let G be a rank two group containing independent elements x' and y'. Let $\alpha_0(p) = \min \{H_p(x'), H_p(y')\}$ for every p. Let $(a_1, b_1) = (1, 0)$, $(a_2, b_2) = (0, 1)$ and let (a_n, b_n) be the n^{th} member of a list of all coprime pairs (a, b) with $a \geqq 0$. Let τ_n be the type of $a_n x' + b_n y'$. Then $\tau_1, \tau_2, \ldots$ is a type sequence of G, and is an $[\alpha_0]$-sequence. Choose characteristics α_n in τ_n so that $\alpha_1 = H(x')$, $\alpha_2 = H(y')$; for

all p, $\alpha_0(p)=\min\{H_p x', H_p y'\}$; if $\alpha_n(p)>\alpha_0(p)$ then $H_p(a_n x'+b_n y')=\alpha_n(p)$; for all $k<n$ and all p, $\min\{\alpha_k(p), \alpha_n(p)\}=\alpha_0(p)$ unless $D(k, n, p)$. Let x and y be independent reals in (0, 1). Let p be a prime. There are five cases; the first four are just as in Theorem 2 of [3].

1. For every n, $\alpha_n(p)=\alpha_0(p)$:

a) $\alpha_0(p)=\infty$. Set $u_p=v_p=0$.

b) $\alpha_0(p)<\infty$. Choose u_p and v_p independent in $I(p)$ with

$$u_p=p^{\alpha_0(p)}(L(p, x)+\ldots),$$

$$v_p=p^{\alpha_0(p)}(L(p, y)+\ldots).$$

2. For exactly one index n, $\alpha_n(p)>\alpha_0(p)$:

a) $\alpha_n(p)<\infty$. Choose independent p-adic integers u_p and v_p satisfying:

$$\alpha_0(p)=\min\{H_p(u_p), H_p(v_p)\} \quad \text{and} \quad H_p(a_n u_p+b_n v_p)=\alpha_n(p).$$

These heights are p-adic values.

b) $\alpha_n(p)=\infty$. Same as (a) except for the independence requirement, which cannot hold now, since $a_n u_p+b_n v_p=0$.

3. For exactly two indices, say $i<j$, $\alpha_0(p)<\alpha_i(p)$, $\alpha_0(p)<\alpha_j(p)$. Then $D(i, j, p)$. By Lemma 4, $\alpha_i(p)=\alpha_0(p)+H_p \det(c_i, c_j)$, so there exist p-adic integers u_p, v_p with $H_p(a_k u_p+b_k v_p)=\alpha_k(p)$ for $k=i$, $k=j$, (note that $a_j u_p+b_j v_p=0$) and $\min\{H_p(u_p), H_p(v_p)\}=\alpha_0(p)$.

Now we have a group $A(x, y; u, v)=A$ of rank two . We claim that $a_n x'+b_n y'$ and $a_n x+b_n y$ have the same type (the first computed in G and the second in A). First consider infinity places. If $\alpha_j(p)=\infty$ then p satisfies (1a), (2b) or (3), and in each of these cases $H_p(a_j x+b_j y)=\infty$. Suppose conversely that $H_p(a_j x+b_j y)=\infty$. This means that $a_j u_p+b_j v_p=0$ so (1a), (2b) or (3) holds for p. Case (1a) gives $\alpha_j(p)=\infty$. Failure of case (1a) implies that for some m, $\alpha_m(p)=\infty$, $a_m u_p+b_m v_p=0$. But not both u_p and v_p are zero. Hence $a_m b_j-a_j b_m=0$, $m=j$. Again $\alpha_j(p)=\infty$. Next consider finite places. The construction makes $\alpha_0(p)=\min\{H_p(x), H_p(y)\}$, for all p; $H_p(a_n x+b_n y)\geqq\alpha_n(p)$ for all p, and if $\alpha_n(p)>\alpha_0(p)$ then equality holds. Hence we need merely to deduce a contradiction from the following assumption: there is an infinite set P of primes and a fixed index k such that for all p in P,

$$H_p(a_k x+b_k y)>\alpha_k(p)=\alpha_0(p)=H_p(a_k x'+b_k y').$$

(Note that $\alpha_k(p)=H_p(a_k x'+b_k y')$ is valid for almost all primes p.) The construction in case (1) guarantees that for almost all p in P there exists n with $\alpha_n(p)>\alpha_0(p)$. Then $H_p(a_n x+b_n y)=\alpha_n(p)$; application of Lemma 3 (to A) shows that p divides $a_n b_k-a_k b_n$. But $H_p(a_n x'+b_n y')\geqq\alpha_n(p)>\alpha_0(p)$ so $H_p(a_k x'+b_k y')>\alpha_0(p)$ by the same Lemma applied to G. This is a contradiction. We have proved:

Theorem 2. *Every type sequence of a rank two torsionfree Abelian group is a type sequence of some group $A(x, y; u, v)$ of rank two.*

Let $A(x, y; u, v)$ be arbitrary. Then any type sequence of A is an $[\alpha_0]$-sequence where α_0 is the g.l.b. of the heights of x and y. Let x' and y' be independent reals. We define u' and v' as follows: if $\alpha_0(p)<\infty$ set $u'_p=p^{-\alpha_0(p)}u_p$, $v'_p=p^{-\alpha_0(p)}v_p$; if $\alpha_0(p)=\infty$ take u'_p and v'_p independent with coefficients of p^0 equal to $L(p, x')$, $L(p, y')$, respectively. If (a, b) is any ordered coprime pair, and if $\alpha_0(p)<\infty$,

then $H_p(ax'+by')=H_p(ax+by)-\alpha_0(p)$; if $\alpha_0(p)=\infty$ then $H_p(ax'+by')<\infty$ by independence of u_p' and v_p' while $H_p(ax'+by')$ is zero for almost all such primes p. Hence the type of $ax'+by'$ is always equal to $T(ax+by)-[\alpha_0]$. The type sequence of $A(x', y', u', v')$ is therefore a zero-sequence, where zero is the type that is identically zero. This proves:

Theorem 3. *A τ_0-sequence is a type sequence of a rank two group if and only if the zero-sequence obtained by subtracting τ_0 from every term is likewise a type sequence of a rank two group.*

Next we prove some results concerning quotient divisible (q.d.) groups. See BEAUMONT and PIERCE [1], and also [3], Corollary 3.

Let G be a q. d. group containing a free group F on the base X (X is linearly independent), with G/F a divisible torsion group. Then for every prime p, min $\{H_p(x); x\in X\}$ is either zero or infinity. For if $0<\min\{H_p(x); x\in X\}<\infty$ and $H_p(x_1)=h<\infty$, for x_1 in X, then $p^{-h}x_1$ belongs to G, has height zero at p, and hence for every member y of F, $H_p(p^{-h}x_1+y)=0$. This contradicts the divisibility of the coset of $p^{-h}x_1$ by p.

For any finite rank torsionfree group G, let $\tau(G)$ be the g.l.b. of the types of any maximal independent set in G. A corollary of the preceding paragraph is that *if G is a q.d. group of finite rank then $\tau(G)$ is non-nil.* This is Theorem 6. 4 of [6].

Let $A(S; x)$ be arbitrary and let $\alpha(p)=\min\{H_p(s); s\in S\}=\min\{H_p(x(s, p))$; $s\in S\}$, for every p. *If α is non-nil then $A(S; x)$ is q.d.* To prove this set

$$m=\prod\{p^{-\alpha(p)}; \alpha(p)<\infty\}.$$

This is a rational since α is non-nil. For every s, ms belongs to $A=A(S; x)$, and for every p, either every $mx(s, p)$ is zero or else at least one of them is a unit in $I(p)$. Let F be the free group generated in A by all ms. Let $a=\sum a_s s$ belong to A and let p be a prime. The proof is completed by showing that the coset of a (mod F) is divisible by p. There is a t in S such that $mx(t, p)$ is a unit in $I(p)$, while mt belongs to F. There is a rational integer k such that $\sum a_s x(s, p)-kmx(t, p)$ is divisible by p in $I(p)$ whence $a-kmt$ is divisible by p in A. Thus the coset of a is divisible by p.

For the special case of rank two groups, the foregoing, together with Theorem 2, implies: *the type set of a rank two group G is the type set of some q.d. group if and only if $\tau(G)$ is non-nil.* Compare [3], Corollary 3.

Next we shall prove a sufficiency theorem, for the rank two case only, which improves our Theorem 2 of [3].

Let S be a zero-sequence, that is, a τ_0-sequence where τ_0 contains the characteristic that is everywhere zero, and let f map S onto S' by permuting indices. Then S' is also a zero-sequence. Moreover, the map f carries free subsequences to free subsequences, snarls to snarls, and if S is a type sequence of a rank two group then so is S'. Given that S contains a free subsequence we may assume, for purposes of showing that S is type sequence of a rank two group, without losing generality, that (we are still assuming that S is a zero-sequence)

A. The subsequence $\tau_1, \tau_3, \ldots$ is free; if $\tau_n(p)=\infty$ then $p\geqq p_{[\frac{1}{2}n]}$ (p_k is the k^{th} prime).

We simplify the problem further by choice of characteristics α_n in τ_n; each step involves, as is easily verified, only putting a finite number of finite places equal to zero.

B. For all p, $\min\{\alpha_1(p), \alpha_2(p)\}=0$; if $k<n$ then $\min\{\alpha_k(p), \alpha_n(p)\}=0$ unless $\alpha_n(p)=\infty$.

C. If n is odd and $k<n$, then not $D(k, n, p)$.

D. If $k<n$ and $D(k, n, p)$, then α_k is a snarl. It is now true that from $D(k, n, p)$ follows: $k<n$, α_k is a snarl, and n is even.

E. The set $F=\{n;$ for all k and all p, not $D(k, n, p)\}$ contains all odd indices, and the set of all α_n for n in F is free.

We make a further assumption that reduces the generality very much; namely that *the set consisting of all snarls and all very large elements is free*, where a characteristic is *very large* if and only if it is infinite at infinitely many primes. Then there is no further loss of generality in assuming:

F. If α_n is finite at every prime, or a snarl, or very large, or if n is odd, then n belongs to F.

Now we shall index the coprime pairs. Let $c=(a, b)$, $c'=(a', b')$; then $\det(c, c')$ is, by definition, $ab'-a'b$. Take $c_1=(1, 0)$. Let $n>1$. If n belongs to F let c_n be the first member of the standard list that has not been previously chosen. Let n be outside F. Take c_n as the first not previously chosen member of the standard list that is a solution for c to the problem: If $\alpha_n(p)=\infty$ then

$$H_p \det(c, c_i)=\alpha_i(p), \quad \text{if} \quad D(i, n, p);$$
$$H_p \det(c, c_{n-1})=1, \quad \text{if} \quad \alpha_i(p)=0 \quad \text{for all} \quad i\neq n.$$

The first case arises only for $i<n$ so our equations refer always to previously chosen pairs. The first case can arise only for finitely many p and involves only one equation for each such p, simply because the type sequence is a zero-sequence. The second case arises only for finitely many p because of our assumption that if α_n is very large then n belongs to F. The Lemma following the next Theorem states that this system is solvable. If $\alpha_n(p)=\infty$ then there is exactly one $h=h(n, p)$ such that (*a*) $h\leqq n$; (b) $\det(c_h, c_n)$ is divisible by p; (*c*) if n belongs to F then $h=n$; (d) if $D(i, n, p)$ then $h=i<n$; (e) if n is not in F and $\alpha_i(p)=0$ for all $i\neq n$, then $h=n-1$. Then (f) if $h<n-1$, then $\alpha_h(p)>0$; and (g) h belongs to F. To see (g), note that if case (d) holds then α_i is a snarl by (D) so by (F), α_i belongs to F. If condition (e) holds then n is even, $n-1=h$ is odd and so belongs to F. Before constructing the group $A(x, y; u, v)$ we make one further adjustment of the characteristics:

G. If $\alpha_n(p)<\infty$, if $\alpha_i(p)=0$ for all $i\neq n$, and if p divides $\det(c_i, c_n)$ for some i less than n, then $\alpha_n(p)=0$.

Note that with this change, α_n has the same type and that the recursion relations for the c_n are not affected. The recursion relations on the c_n enable us to choose u and v, given independent reals x and y in (0, 1), just as in Theorem 2, to get a group $A=\mathrm{A}(x, y; u, v)$. We shall show that for every n, type of $a_n x+b_n y$ is $[\alpha_n]$. For infinity places the argument of Theorem 2 is valid. For finite places we need merely to derive a contradiction from the assumption: there is an index k and an infinite set Q of primes such that for all p in Q, $0=\alpha_k(p)<H_p(a_k x+b_k y)$. As in the previous theorem, if $\alpha_n(p)>0$ then $H_p(a_n x+b_n y)=\alpha_n(p)$, and from our assumption on k and Q, there exists, for every p in Q, an $m=m(p)>k$ with $\alpha_m(p)>0$. Hence $H_p(a_m x+b_m y)=$ $=\alpha_m(p)>0$ and therefore p divides $\det(c_m, c_k)$. By condition (G), there exists $n=n(p)\geqq m(p)$ with $\alpha_n(p)=\infty$. Then p divides $\det(c_n, c_k)$. The set of all $m(p)$

and the set of all $n(p)$, for p in Q, are both infinite sets; for example, p divides $\det(c_{n(p)}, c_k)$ for all p in the infinite set Q, while $n(p)>k$. Let $h=h(p)=h(n(p), p)$ (see remarks preceding condition (G)), so that h belongs to F. By (b), p divides $\det(c_n, c_h)$ and therefore p divides $\det(c_h, c_k)$. Suppose that the set of all $h(p)$, p in Q, is finite. Then there is a j with $h(p)=j$ for infinitely many p in Q, and for all these values of p we have $h(p)=h(n(p), p)<n(p)-1$. By condition (f), $\alpha_j(p)>0$, so p divides $\det(c_k, c_{h(p)})=\det(c_k, c_j)$ for infinitely many values of p and so $j=k$. This gives $0=\alpha_k(p)=\alpha_j(p)>0$, a contradiction. Hence the set of all $h(p)$ is infinite. Now taking into account that $h(p)$ belongs to F the argument of Theorem 2 of [3] is essentially applicable: for all large $h(p)=h$,

$$0<|\det(c_k, c_h)|<Bh(p)<B'h\log h<p_{[\frac{1}{2}h]}\leqq p_{[\frac{1}{2}n]},$$

where B and B' are fixed positive numbers. But $\alpha_n(p)=\infty$ so by (A), $p\geqq p_{[\frac{1}{2}n]}$, while p divides $\det(c_k, c_{h(p)})$. This is another contradiction and completes the proof of:

Theorem 4. *If the τ_0-sequence S has an infinite free subsequence and if the set of all snarls and very large elements is free, then S is a type sequence of a rank two torsionfree Abelian group.*

Lemma. *Let (a_i, b_i) be a coprime pair, e_i a non-negative integer, p_i a prime, $i=1, \ldots, n$, with the p_i distinct. Then there exist infinitely many coprime pairs (x, y) such that for every i,*

$$H_{p_i}(a_ix-b_iy)=e_i.$$

PROOF. Choose s_i and t_i so that $s_ia_i-t_ib_i=1$, and construct u and v so that, mod $p_i^{e_i+1}$, (use Chinese Remainder Theorem), for every i,

$$u\equiv s_ip_i^{e_i}+b_i, \qquad v\equiv t_ip_i^{e_i}+a_i.$$

Let g be the g.c.d. of u and v with $u=gx_0$, $v=gy_0$. Then (x_0, y_0) is a solution. Let p be different from all the p_i and let (x_j, y_j) be a coprime pair satisfying $H_{p_i}(a_ix_j-b_iy_j)=e_i$, $i=1, \ldots, n$ and $H_p(a_ix_j-b_iy_j)=j$, $j=0, 1, 2, \ldots$. Then the (x_j, y_j) are infinite in number, and all are solutions.

Example. Let α_1 be one at every odd numbered prime and zero at every even numbered prime. For even n let α_n be zero at every prime. For odd $n\geqq 3$, set α_n zero at every prime except the n^{th}, with $\alpha_n(p_n)=\infty$. According to Theorem 4, the sequence $[\alpha_1]$, $[\alpha_2]$, ... is a type sequence of a group $A(x, y; u, v)$; i.e., for a suitable indexing of the coprime pairs $c_n=(a_n, b_n)$, $a_n\geqq 0$, the type of a_nx+b_ny is $[\alpha_n]$. For each odd $n\geqq 3$, let $k_n=\max\{p;$ for some $i<n$, p divides $\det(c_i, c_n)\}$. Let $\beta_1=\alpha_1$, and $\beta_n=\alpha_n$ for all even n. Let $\beta_3(p_3)=\infty$, set

$$Q_3=\{p;\ p>k_3 \quad \text{and for all} \quad i<3,\ \beta_i(p)=0\},$$

and set β_3 equal to ∞ at the first, third, fifth, etc., primes in Q_3, zero elsewhere. Having defined β_k for all $k<n$, where n is odd, and having defined Q_k for all odd $k<n$, set $\beta_n(p_n)=\infty$, set

$$Q_n=\{p;\ p>k_n \quad \text{and for all} \quad i<n,\ \beta_i(p)=0\},$$

and set β_n equal to ∞ at the first, third, fifth, etc., primes in Q_n, zero elsewhere. The sequence $[\beta_1]$, $[\beta_2]$, ... is a zero-sequence, $[\beta_n]$ is very large for all odd $n \geqq 3$, while β_1 is a snarl of the set of all these very large elements. With c_n defined as in constructing $A(x, y; u, v)$ we can construct a group $A(x', y'; u', v')$ exactly as in Theorem 4, so that the type of $a_n x' + b_n y'$ is $[\beta_n]$. Thus the condition that the set of all very large elements be free is not necessary.

We close with an alternate formulation of the rank two problem, suggested by our various sufficiency theorems, especially Theorem 2. Let $\alpha: \alpha_1, \alpha_2, \ldots$ be a sequence of characteristics such that the corresponding sequence S of types is a zero-sequence. Call α a *solvable sequence* provided there exists an indexing $c_1, c_2, \ldots$ of the coprime pairs with nonnegative first member so that for every i there exists an N such that for every prime p and every index $n > N$, if $\alpha_n(p) = \infty$ then

$$H_p \det (c_i, c_n) = \alpha_i(p).$$

Then we can prove, by methods almost identical with those used previously, that *S is a type sequence of a rank two group if and only if α is a solvable sequence.*

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