

On Hausdorff and related moment problems

By A. JAKIMOVSKI, (Tel-Aviv) D. LEVIATAN (Tel-Aviv) and M. S. RAMANUJAN*
(Ann Arbor, Mich).

The Hausdorff moment problem relates to obtaining necessary and sufficient conditions involving a real sequence $\{\lambda_n\}$ so that there exists a function $g(t)$ of bounded variation over $[0, 1]$ and $\lambda_n = \int_0^1 t^n dg(t)$. Different solutions of this problem have been given by HAUSDORFF [6], RAMANUJAN [10] and JAKIMOVSKI [7]. A related question is to obtain necessary and sufficient conditions for the representation $\lambda_n = \int_0^1 t^n g(t) dt$, with $g(t)$ belonging to a specified function space. This has been discussed in HAUSDORFF [6], RAMANUJAN [12], BERMAN [1, 2], LORENTZ [9] and GEHRING [4]. The above problems with the modifications of the type that $\lambda_n = \int_0^1 t^{n+\alpha} dg(t)$ or $\lambda_n = \int_0^1 t^{n+\alpha} g(t) dt$ form the content of [3] and [8]. The main tool in most of these investigations is the uniform approximation of continuous functions in $[0, 1]$ by polynomials or power series suggested by them. In this paper we obtain certain sufficient conditions for the above representations under a fairly general set-up and indicate how some of the earlier results are included as special cases of these results.

Notation. Throughout the paper the following assumptions are made, unless a statement to the contrary is made.

1. $R > 1$ is a fixed real number;
2. $\varphi(n)$ denotes a sequence of non-negative integers or $+\infty$;
3. For each fixed n , ($n=0, 1, 2, 3, \dots$), $\{\alpha_{nm}\}$ is a sequence of real numbers such that $\alpha_{nm}=0$, for $m > \varphi(n)$ and $0 \leq \alpha_{nm} \leq 1$ for all n and m ;
4. $c_{nm}(z)$ are power-series satisfying

$$(4.1) \quad c_{nm}(z) = 0 \quad \text{for } m > \varphi(n), \quad n = 0, 1, 2, \dots;$$

$$(4.2) \quad \text{for } c_{nm}(z) = \sum_{k=0}^{\infty} a_{nmk} z^k, \quad (0 \leq m \leq \varphi(n)), \quad \text{we have } \sum_{k=0}^{\infty} |a_{nmk}| R^k < \infty$$

$$(4.3) \quad \text{for each } p = 0, 1, 2, 3, \dots, A_{np}(z) = \sum_{m=0}^{\infty} c_{nm}(z) \alpha_{nm}^p \rightarrow z^p$$

uniformly in $|z| \leq R$, as $n \rightarrow \infty$;

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5. In the space P of power series $g(z) = \sum_0^\infty a_n z^n$, with radius of convergence greater than 1, set $L(g) = \sum_0^\infty a_n \lambda_n$, where $\{\lambda_n\}$ is a fixed bounded sequence. $L(g)$ is well defined and is a linear functional on P .

We list below some special examples of functions described on (4).

Example 1. $c_{nm}(z) = \binom{n}{m} (1-z)^{n-m} z^m$, $0 \leq m \leq n$ and $= 0$ for $m > \varphi(n) = n$ and $\alpha_{nm} = m/n$.

The function described above is the kernel of the well-known Bernstein approximation for analytic functions; see, for instance, Lorentz ([9], Theorem 4.1.1).

Example 2. $c_{nm}(z) = \binom{n}{m} [z + Q_n(z)]^m [1 - z - Q_n(z)]^{n-m}$ for $0 \leq m \leq n$ and $= 0$, $m > \varphi(n) = n$ where for each n , $|z + Q_n(z)| \leq R$ for $|z| \leq R$, $R > 1$ being fixed, α_{nm} are chosen as in Example 1, and $Q_n(z) = \sum_r q_{nr} z^r$ with $\varepsilon_n = \sum_r |q_{nr}| R^r \rightarrow 0$ as $n \rightarrow \infty$.

Example 3. The Lagrange interpolatory functions.

Example 4. In Example 2, choose $Q_n(z) = \frac{a_n(1-z)}{1+a_n}$, with $a_n > 0$ and $a_n \rightarrow 0$.

Theorems

In this section we prove certain sufficient conditions for a given sequence $\{\lambda_n\}$ to be a Hausdorff moment sequence. We start with the following lemma.

Lemma 1. Suppose $R > 1$; if $f_n(z) = \sum_{k=0}^\infty a_{nk} z^k \rightarrow z^p$ uniformly in $|z| \leq R$ and if $\{\lambda_n\}$ is a bounded sequence, then $\lim_{n \rightarrow \infty} L(f_n) = \lambda_p$.

PROOF. We first prove that for each $\varepsilon > 0$ there corresponds a $N(\varepsilon)$ such that for $n > N(\varepsilon)$, we have $|a_{nk} - \delta_{kp}| < \varepsilon R^{-k}$, where δ_{kp} are the Kronecker symbols.

For any $k \geq 0$,

$$a_{nk} = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f_n(z)}{z^{k+1}} dz$$

and

$$\delta_{kp} = \frac{1}{2\pi i} \oint_{|z|=R} \frac{z^p}{z^{k+1}} dz.$$

Consequently

$$|a_{nk} - \delta_{kp}| = \left| \frac{1}{2\pi i} \oint_{|z|=R} \frac{f_n(z) - z^p}{z^{k+1}} dz \right| \leq \max_{|z|=R} \frac{|f_n(z) - z^p|}{R^k}$$

and the conclusion above easily follows.

Now to complete the proof of the lemma, consider any $\varepsilon > 0$ and the corresponding $N(\varepsilon)$ indicated above. For $n > N(\varepsilon)$,

$$\begin{aligned} |L(f_n) - \lambda_p| &= \left| \sum_k a_{nk} \lambda_k - \lambda_p \right| \\ &= \left| \sum_k (a_{nk} - \delta_{kp}) \lambda_k \right| \\ &\leq \sup_k |\lambda_k| \sum_k |a_{nk} - \delta_{kp}| \leq \sup_k |\lambda_k| \varepsilon \sum_k R^{-k} \end{aligned}$$

and the proof of the lemma is complete.

We state a theorem below and omit its proof remarking that it runs on lines parallel to that of a known theorem (JAKIMOVSKI [7], Theorem 1) and that Lemma 1 plays now the rôle of Theorem 3 of [7].

Theorem 1. *Let $\varphi(n)$ be finite for $n \geq 0$. If for the bounded sequence $\{\lambda_n\}$ we have*

$$(1.1) \quad \sup_n \sum_{k=0}^{\infty} |b_{nk}| \sum_{m=0}^{\varphi(k)} |L(c_{km})| < \infty$$

for some regular matrix (i. e., Toeplitz matrix) $B = (b_{nk})$, then $\{\lambda_n\}$ is a Hausdorff moment-sequence.

The following remarks on Theorem 1 are relevant.

1. The following converse of Theorem 1 is easily proved: If $\sup_n \max_{0 \leq z \leq 1} \sum_{m=0}^{\varphi(n)} |c_{nm}(z)| < \infty$ and $\{\lambda_n\}$ is a Hausdorff moment sequence then for *each* regular matrix $B = (b_{nk})$, we have that (1.1) holds.

2. Theorem 1 of [7] referred to earlier is itself a special case of our Theorem 1 by choosing the $c_{nm}(z)$ and α_{nm} as in Example 1.

3. In the case of Example 2, the condition (1.1) becomes

$$\sup_n \sum_{k=0}^{\infty} |b_{nk}| \sum_{m=0}^n \binom{n}{m} |L([z + Q_n(z)]^m [1 - z - Q_n(z)]^{n-m})| < \infty$$

and considering Example 4, it follows that a bounded sequence $\{\lambda_n\}$ will be a Hausdorff moment sequence if, in particular, $a_n > 0$, $a_n \rightarrow 0$ and

$$\sup_n (1 + a_n)^{-n} \sum_{m=0}^n \binom{n}{m} \left| \sum_{r=0}^m \binom{m}{r} a_n^{m-r} \Delta^{n-m} \lambda_r \right| < \infty.$$

Indeed the above condition is also seen to be necessary.

4. It is no serious limitation to assume that the sequence $\{\lambda_n\}$ is bounded since a moment sequence is necessarily bounded.

Theorem 2. *Let $\varphi(n) = O(n)$ and let for each fixed n , $\{c_{nM}(z)\}$, $M = 0, 1, \dots, \varphi(n)$, denote some permutation of $\{c_{nm}(z)\}$, $m = 0, 1, 2, \dots, \varphi(n)$. If for a bounded sequence $\{\lambda_n\}$,*

$$(2.1) \quad \sup_M \sum_{n=0}^{\infty} |L(c_{nM})| < \infty$$

then $\{\lambda_n\}$ is a Hausdorff moment sequence.

PROOF. Let

$$\sup_M \sum_{n=0}^{\infty} |L(c_{nM})| = K.$$

For each fixed n , let $N_n = \max_{j=0, 1, \dots, n} \varphi(j)$. Then

$$(N_n + 1)K \cong \sum_{M=0}^{N_n} \sum_{k=0}^{\infty} |L(c_{kM})| = \sum_{k=0}^{\infty} \sum_{M=0}^{N_n} |L(c_{kM})| \cong \sum_{k=0}^n \sum_{M=0}^{\varphi(k)} |L(c_{kM})|.$$

Thus, for each fixed n ,

$$\begin{aligned} \infty > K^* &= \frac{(N_n + 1)K}{(n + 1)} \cong \frac{1}{n + 1} \sum_{k=0}^n \sum_{M=0}^{\varphi(k)} |L(c_{kM})| = \\ &= \sum_{k=0}^n \frac{1}{n + 1} \sum_{M=0}^{\varphi(k)} |L(c_{kM})| = \sum_{k=0}^n \frac{1}{n + 1} \sum_{m=0}^{\varphi(k)} |L(c_{km})| \end{aligned}$$

and the result follows by taking the matrix $\mathbf{B}$ in Theorem 1 to be that of the $(C, 1)$ method.

Corollary and Remark. Choosing the $c_{nm}(z)$ as in Example 1 it follows that

$$(2.2) \quad \sup_n \sum_{m=0}^{\infty} \left| \binom{n}{m} \Delta^{n-m} \lambda_m \right| < \infty$$

implies that $\{\lambda_n\}$ is necessarily a moment sequence, thus enabling us to make the following remark on Theorem 219 of HARDY [5]. If the quasi-Hausdorff method (H^*, μ_n) is conservative (i. e.) if it is a Kojima matrix, then it is necessary that μ_n is a moment sequence. (see also RAMANUJAN [11], p. 201).

We consider next the case the $\varphi(n)$ is not necessarily finite for any n and prove, with the notations set out earlier,

Theorem 3. Let $\varphi(n)$ be not necessarily finite for any n . Suppose also the following conditions hold:

$$(i) \quad \text{for each } p \cong 0, \quad \sum_{m=0}^{\varphi(n)} |\alpha_{nm}|^p \sum_{k=0}^{\infty} |a_{nmk}| < \infty$$

and

$$(ii) \quad \text{for } \{\lambda_n\} \text{ bounded, } \sup_n \sum_{m=0}^{\varphi(n)} |L(c_{nm})| < \infty.$$

Then $\{\lambda_n\}$ is a Hausdorff moment sequence.

PROOF. We first prove that under the assumptions of the theorem

$$(3.1) \quad \lim_{n \rightarrow \infty} \sum_{m=0}^{\varphi(n)} L(c_{nm}) \alpha_{nm}^p = \lambda_p, \quad \text{for each } p \cong 0.$$

Let p be fixed. The uniform convergence, in $|z| \leq R$, of $\sum_{m=0}^{q(n)} c_{nm}(z) \alpha_{nm}^p$ to z^p implies, by the Wierstrass's theorem for double series, that

$$A_{np}(z) = \sum_{m=0}^{q(n)} c_{nm}(z) \alpha_{nm}^p = \sum_{j=0}^{\infty} b_{npj} z^j,$$

where

$$b_{npk} = \sum_{m=0}^{q(n)} a_{nmk} (\alpha_{nm})^p.$$

We now have $A_{np}(z) \rightarrow z^p$ uniformly in $|z| \leq R$, as $n \rightarrow \infty$. Therefore, by Lemma 1,

$$(3.2) \quad \lim_{n \rightarrow \infty} L(A_{np}) = \lim_{n \rightarrow \infty} \sum_{j=0}^{\infty} b_{npj} \lambda_j = \lambda_p.$$

This is true for $p=0, 1, 2, 3, \dots$.

Now

$$\sum_{m=0}^{q(n)} L(c_{nm}) \alpha_{nm}^p = \sum_{m=0}^{q(n)} \alpha_{nm}^p \sum_{k=0}^{\infty} a_{nmk} \lambda_k$$

and

$$\sum_{m=0}^{q(n)} |\alpha_{nm}|^p \sum_{k=0}^{\infty} |a_{nmk}| |\lambda_k| \leq \sup_k |\lambda_k| \sum_{m=0}^{q(n)} |\alpha_{nm}|^p \sum_{k=0}^{\infty} |a_{nmk}| < \infty$$

and therefore interchanging the order of summation we have,

$$\sum_{m=0}^{q(n)} L(c_{nm}) \alpha_{nm}^p = \sum_{k=0}^{\infty} \lambda_k \sum_{m=0}^{q(n)} \alpha_{nm}^p a_{nmk} = \sum_{k=0}^{\infty} b_{npk} \lambda_k$$

and therefore (3.1) follows from (3.2).

It follows from (3.1) that for each polynomial $P(z)$ we have

$$\lim_{n \rightarrow \infty} \sum_{m=0}^{q(n)} L(c_{nm}) P(\alpha_{nm}) = L(P).$$

Also,

$$\left| \sum_{m=0}^{q(n)} L(c_{nm}) P(\alpha_{nm}) \right| \leq \|P\| \sum_{m=0}^{q(n)} |L(c_{nm})| = K \|P\|,$$

for each n , the norm above being the supremum norm over $[0, 1]$. Letting $n \rightarrow \infty$, it follows that for each polynomial P over $[0, 1]$, $|L(P)| \leq K \|P\|$; i. e. L is a linear continuous functional over $P[0, 1]$, the space of polynomials over $[0, 1]$. Since $P[0, 1]$ is dense in $C[0, 1]$, L can be extended to $C[0, 1]$ and the result in the theorem is now an easy consequence of the Riesz representation theorem.

Our final result in Theorem 4 is a sufficient condition for a bounded sequence

$\{\lambda_n\}$ to be a moment sequence of the form $\lambda_n = \int_0^1 t^n f(t) dt$, where f is in a specified function-space $X(C)$ of the Koethe type. For definition and relevant details regarding these see LORENTZ ([9], pp. 65—69).

Theorem 4. Let the function-space $X(C)$ have the property of rearrangement-invariant norm and uniform absolute continuity for functions with norm less than 1. Let $\varphi(n) < \infty$, for each n . Let $0 = \alpha_{n0} < \alpha_{n1} < \alpha_{n2} < \dots < \alpha_{n, \varphi(n)} = 1$ and $\sup_m |\alpha_{nm} - \alpha_{n, m-1}| \rightarrow 0$ as $n \rightarrow \infty$. Given a bounded sequence $\{\lambda_n\}$, define the functions $f_n(x)$ as follows:

$$f_n(x) = (\alpha_{nm} - \alpha_{n, m-1})^{-1} L(c_{nm}), \quad \alpha_{n, m-1} \leq x < \alpha_{nm}, \quad m = 0, 1, 2, \dots, \varphi(n).$$

Then, if $\|f_n\|_{X(C)} \leq M$, for each fixed n , then $\lambda_n = \int_0^1 t^n f(t) dt$, for a suitable function f in the space $X(C)$, with $\|f\|_{X(C)} \leq M$.

PARTIAL PROOF OF THEOREM 4. We have, in the notation set out earlier, $\sum_{m=0}^{\varphi(n)} c_{nm}(z) \alpha_{nm}^p \rightarrow z^p$, uniformly in $|z| \leq R$, $R > 1$,

$$\text{i. e.} \quad \sum_{m=0}^{\varphi(n)} \alpha_{nm}^p \sum_{k=0}^{\infty} a_{nmk} z^k = \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\varphi(n)} a_{nmk} \alpha_{nm}^p = \sum_{k=0}^{\infty} b_{nk}^{(p)} z^k,$$

say, and the above expression converges uniformly in $|z| \leq R$ to z^p , for each p . Therefore, by Lemma 1,

$$\sum_{k=0}^{\infty} b_{nk}^{(p)} \lambda_k \rightarrow \lambda_p \quad \text{for each } p, \quad \text{as } n \rightarrow \infty;$$

i. e.

$$\sum_{k=0}^{\infty} \lambda_k \sum_{m=0}^{\varphi(n)} a_{nmk} \alpha_{nm}^p = \sum_{m=0}^{\varphi(n)} \alpha_{nm}^p \sum_{k=0}^{\infty} a_{nmk} \lambda_k \rightarrow \lambda_p \quad \text{as } n \rightarrow \infty;$$

i. e.

$$\lim_n \sum_{m=0}^{\varphi(n)} L(c_{nm}) \alpha_{nm}^p = \lambda_p, \quad \text{for } p = 0, 1, 2, 3, \dots$$

But the left-side above is $\int_0^1 f_n(x) g_n(x) dx$, where the $f_n(x)$ are the functions defined in the statement of the theorem and

$$g_n(x) = \alpha_{nm}^p, \quad \text{for } \alpha_{n, m-1} \leq x < \alpha_{nm}, \quad m = 0, 1, \dots, \varphi(n).$$

Hereafter the proof is routine and we refer the reader to the proof of Theorem 3.8.4 in LORENTZ [9].

Corollary. For the space $L_p(p > 1)$, the condition $\|f_n\| \leq M$, reduces to

$$\sum_{m=0}^{\varphi(n)} \frac{|L(C_{nm})|^p}{(\alpha_{nm} - \alpha_{n, m-1})^{p-1}} \leq M, \quad n = 0, 1, \dots$$

The choice of $c_{nm}(z)$ as in Example 1 yields the well-known solution of the moment problem for the space $L_p(p > 1)$.

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