# On the equation $\varphi(\sum x_i) = \sum a_{ij}\varphi(x_i)g(x_i)$

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It is known that the functions $\varphi(x)=Bx$ ($B=$const.) can be characterized as the only solutions having points of continuity of Cauchy's equation

$$\varphi(x+y) = \varphi(x)+\varphi(y).$$

This equation is equivalent to the equation

$$\varphi(x_1+\ldots+x_n) = \sum_{k=1}^{n} \varphi(x_k) \tag{1}$$

(cf. ACZÉL—KIESEWETTER [3]).

A natural generalization of equation (1) is the equation

$$\varphi(x_1+\ldots+x_n) = \sum_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}\,\varphi(x_i)g(x_j), \tag{2}$$

where $\varphi$ and $g$ are unknown, $a_{ij}$ constants.

Can equation (2) serve to a characterization of some type of functions? Yes. For $n\geqq 3$ it is characteristic for the functions $\varphi(x)=Bx+C$ ($B=$const., $C=$const.).

If $n=2$, the solutions of equation (2) can be obtained from those of the equation

$$\varphi(x+y) = \varphi(x)g(y)+\varphi(y)g(x) \tag{3}$$

which was considered e. g., in [1], [4], [5], [6].

By the results of [5] it is easy to show that in the case $n=2$ the only non-zero solutions $\varphi$, $g$ of equation (2) which have at least one point of continuity in common are the following couples of functions

1° $\quad \varphi(x) = Ae^{ax}, \qquad g(x) = \dfrac{1}{a_{12}+a_{21}}e^{ax}$

2° $\quad \varphi(x) = Axe^{ax}, \qquad g(x) = \alpha e^{ax},$

3° $\quad \varphi(x) = Ae^{ax}\sin bx, \qquad g(x) = \alpha e^{ax}\cos bx,$

4° $\quad \varphi(x) = Ae^{ax}\sinh bx, \qquad g(x) = \alpha e^{ax}\cosh bx,$

where $A$, $a$, $b$ ($A\neq 0$, $b\neq 0$) are arbitrary constanst, and $\alpha=\dfrac{1}{a_{12}}=\dfrac{1}{a_{21}}$.

It can be proved that for $n \geqq 3$ the only non-zero solutions of equation (2) which have a point of continuity in common are

1° $$\varphi(x) = Bx + C, \qquad g(x) = \frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}},$$

2° $$\varphi(x) = Bx, \qquad g(x) = \frac{n}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}} = \frac{1}{\sum\limits_{\substack{j=1\\ i\neq j}}^{n} a_{ij}},$$

where $B$, $C$ ($C\neq 0$, $B\neq 0$ in 2°) are arbitrary constants.

Thus the functions $\varphi(x)=Bx+C$ can be defined as the only functions having at least one point of continuity for which there exists a function $g$ continuous at this point, such that $\varphi$, $g$ constitute the solution of equation (2).

Before we solve equation (2), we shall prove some theorems on the regularity of its solutions. We shall restrict ourselves to the case $n \geqq 3$, since for $n=2$ equation (2) can easily be reduced to equation (3) for which adequate theorems were given in [5]. (Notice that the proof of Theorem I remains valid for $n=2$, too).

**Theorem 1.** *If the functions $\varphi, g$ which satisfy equation* (2) *have at least one point of continuity in common and $\varphi(x)\not\equiv 0$, they are continuous everywhere.*

PROOF. Let $a$ be a common point of continuity of the functions $\varphi$ and $g$.

In order to prove that the function $\varphi$ is continuous at an arbitrary point $\xi$, we fix $x_1, \ldots, x_{n-1}$ such that $x_1+\ldots+x_{n-1}=\xi-a$. If $x_n \to a$, $x_1+\ldots+x_n\to\xi$. Since the right side of equation (2) is continuous at $a$, the function $\varphi$ is continuous at $\xi$.

We shall show that the function $g$ too is continuous everywhere. Since $\varphi(x)\not\equiv 0$, it follows from (2) that $\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}\neq 0$ and for some $j$ it must be $\sum\limits_{\substack{i=1\\ i\neq j}}^{n} a_{ij}\neq 0$, Without loss of generality we can assume that $\sum\limits_{i=1}^{n-1} a_{in}\neq 0$. Given $b$ such that $\varphi(b)\neq 0$ we take $x_1=\ldots=x_{n-1}=b$, $x_n=x$. Then (2) can be written as

$$\varphi((n-1)b+x) = \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} a_{ij}\,\varphi(b)g(b) + \sum_{i=1}^{n-1} a_{in}\,\varphi(b)g(x) + \sum_{j=1}^{n-1} a_{nj}\,\varphi(x)g(b)$$

whence

$$g(x) = \frac{\varphi((n-1)b+x) - \sum\limits_{j=1}^{n-1} a_{nj}\, g(b)\varphi(x) - \sum\limits_{\substack{i,j=1\\ i\neq j}}^{n-1} a_{ij}\,\varphi(b)g(b)}{\sum\limits_{i=1}^{n-1} a_{in}\,\varphi(b)} \tag{4}$$

and it follows from the continuity of the function $\varphi$ that the function $g$ is continuous everywhere.

Now we can make use of Aczél's method (cf. [2]) and prove the following

**Theorem 2.** *If the functions $\varphi$, $g$ satisfy equation (2), are continuous, and moreover $\varphi(x)\not\equiv 0$, they are functions of class $C^\infty$.*

PROOF. Consider first the case $\varphi(0)\neq 0$, e. g. $\varphi(0)=1$. Then

$$g(0) = \frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}},$$

Similarly as in the proof of Theorem 1 we can assume that $\sum\limits_{i=1}^{n-1} a_{in}\neq 0$. Substituting $x_1=\ldots=x_{n-1}=0$, $x_n=x$ into (2) we obtain

$$\varphi(x) = \sum_{j=1}^{n-1} a_{nj}\,\varphi(x)g(0) + \sum_{i=1}^{n-1} a_{in}\,g(x) + \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} a_{ij}\,g(0).$$

Hence

(5) $$g(x)=A\varphi(x)+B,$$

where

$$A = \left(1-\frac{\sum\limits_{j=1}^{n-1} a_{nj}}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}}\right)\frac{1}{\sum\limits_{i=1}^{n-1} a_{in}},$$

$$B = -\frac{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n-1} a_{ij}}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}}\,\frac{1}{\sum\limits_{i=1}^{n-1} a_{in}}.$$

Substituting (5) into (2) we obtain

$$\varphi(x_1+\ldots+x_n) = \sum_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}\,\varphi(x_i)\big(A\varphi(x_j)+B\big).$$

Putting $x_1=\ldots=x_{n-1}=s$, $x_n=x$ and integrating by $s$ from $\alpha$ to $\beta$, we obtain

(6) $$\int_\alpha^\beta \varphi\big((n-1)s+x\big)\,ds = C\varphi(x)+D,$$

where

$$C = A\sum_{i=1}^{n-1}(a_{in}+a_{ni})\int_\alpha^\beta \varphi(s)\,ds + B\sum_{j=1}^{n-1} a_{nj}(\beta-\alpha),$$

$$D = \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} a_{ij}\int_\alpha^\beta \varphi(s)\big(A\varphi(s)+B\big)\,ds + B\sum_{i=1}^{n-1} a_{in}\int_\alpha^\beta \varphi(s)\,ds.$$

If $\varphi(x) \not\equiv \text{const}$, one can choose the numbers $\alpha$, $\beta$ such that $C \neq 0$. Then we obtain from (6)

$$\varphi(x) = \frac{1}{C} \int_{\alpha}^{\beta} \varphi\big((n-1)s+x\big)\, ds - \frac{D}{C}.$$

Since the function $\varphi$ is continuous, the right hand side of the last equality is differentiable, hence it follows that the function $\varphi$ is differentiable.

Now, we can differentiate the right hand side of equation (2) with respect to $x_1, \ldots, x_n$, and therefore the function $\varphi$ which appears on the left hand side of equation (2) must have derivatives of order $\leqq n$. From (4) it follows that this is the case also for the function $g$.

Similarly we conclude that the derivatives of order $> n$ exist.

If $\varphi(x) = \text{const}$, the theorem is trivial (in view of (4)).

In the case $\varphi(0) = 0$ we substitute $x_i = x$, $x_j = 0$ for $j = 1, \ldots, n, j \neq i$ and we obtain

$$\varphi(x) = \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij}\, \varphi(x) g(0), \qquad (i = 1, \ldots, n).$$

Hence

$$g(0) = \frac{1}{\sum\limits_{\substack{j=1 \\ i \neq j}}^{n} a_{ij}}, \qquad (i = 1, \ldots, n). \tag{7}$$

Putting $x_k = 0$ for $k = 1, \ldots, n$, $k \neq i$, $k \neq j$ into (2), we obtain after computations

$$a_{ij}\, \varphi(x_i) g(x_j) + a_{ji}\, \varphi(x_j) g(x_i) = \varphi(x_i + x_j) - \sum_{\substack{k=1 \\ k \neq i,\, k \neq j}}^{n} \big(a_{ik}\, \varphi(x_i) + a_{jk}\, \varphi(x_j)\big) g(0).$$

In view of the last equality and in view of (7) we obtain from (2) the equality

$$\varphi(x_1 + \ldots + x_n) = \sum_{\substack{i, j=1 \\ i<j}}^{n} \varphi(x_i + x_j) - (n-2) \sum_{k=1}^{n} \varphi(x_k). \tag{8}$$

Setting in (8) $x_1 = \ldots = x_{n-1} = s$, $x_n = x$ and integrating for $s$ from $\alpha$ to $\beta$, we obtain after computations

$$\varphi(x) = P \int_{\alpha}^{\beta} \varphi\big((n-1)s+x\big)\, ds + Q \int_{\alpha}^{\beta} \varphi(s+x)\, ds + R, \tag{9}$$

where

$$P = \frac{1}{(n-2)(\beta-\alpha)}, \qquad Q = -\frac{n-1}{(n-2)(\beta-\alpha)},$$

$$R = \frac{n-1}{\beta-\alpha} \left[ \int_{\alpha}^{\beta} \varphi(s)\, ds - \frac{1}{2} \int_{\alpha}^{\beta} \varphi(2s)\, ds \right].$$

Since we can differentiate the right hand side of equality (9), the function $\varphi$ must have a first derivative. In view of equality (4) (which was obtained in the proof of Theorem I), we conclude that the first derivative of the function $g$ exists too.

Making use of equation (2) it is easy to show that the second derivative of the function $\varphi$ also exists. The existence of the second derivative of the function $g$ follows automatically from (4).

The proof of the existence of the higher derivatives of these functions is analogous.

As a simple conclusion from theorem 1 and 2 we get the following

*Remark*. If the functions $\varphi$, $g$ which satisfy equation (2) have at least one point of continuity in common and $\varphi(x) \not\equiv 0$, they are functions of class $C^\infty$.

Now, one can easily find the solutions $\varphi$, $g$ of equation (2) which have a point of continuity in common.

**Theorem 3.** *The solutions $\varphi, g$ of equation* (2) *with $n \geqq 3$ which have a point of continuity in common, and such that $\varphi(x) \not\equiv 0$, must have the form*

$$1^\circ \qquad \varphi(x) = Bx + C, \qquad g(x) = \frac{1}{\sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij}},$$

or

$$2^\circ \qquad \varphi(x) = Bx, \qquad g(x) = \frac{1}{\sum\limits_{\substack{j=1 \\ i \neq j}}^{n} a_{ij}}, \qquad (i = 1, \ldots, n),$$

*where $B$, $C$ ($C \neq 0$, $B \neq 0$ in $2^\circ$ are arbitrary constants.*

Proof. By the Remark the functions $\varphi$ and $g$ have all the derivatives. Differentiating equation (2) for three different variables, we obtain $\varphi'''(x) \equiv 0$. Hence it follows that the function $\varphi$ has the form

$$\varphi(x) = Ax^2 + Bx + C. \tag{10}$$

Differentiating equation (2) by $x_i$ and $x_j$ ($i \neq j$), we obtain

$$\varphi''(x_1 + \ldots + x_n) = a_{ij} \varphi'(x_i) g'(x_j) + a_{ji} \varphi'(x_j) g'(x_i).$$

Hence, in view of (10), we have

$$2A = a_{ij}(2Ax_i + B) g'(x_j) + a_{ji}(2Ax_j + B) g'(x_i). \tag{11}$$

Differentiating (11) by $x_i$ and $x_j$ ($i \neq j$) and putting $x_i = x_j = x$, we obtain

$$(a_{ij} + a_{ji}) A g''(x) = 0.$$

It must be $Ag''(x) = 0$, since in the contrary case we would have $a_{ij} + a_{ji} = 0$ for $i, j = 1, \ldots, n$, $i \neq j$, and then (in view of (2)) $\varphi(x) \equiv 0$.

Now, we shall prove that $g''(x)\equiv 0$. In fact, $Ag''(x)=0$ implies either $g''(x)\equiv 0$ or $A=0$. If $A=0$, it follows from (11) that $(a_{ij}+a_{ji})Bg'(x)=0$ $(i, j=1, \ldots, n, i\neq j)$. Since the assumption $Bg'(x)\neq 0$ yields $\varphi(x)\equiv 0$, it must be either $B=0$ or $g'(x)\equiv 0$. We shall prove that $g'(x)\equiv 0$. Indeed, if $B=0$, then $\varphi(x)=C$ and it follows from (2) that

$$g(x) = \frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}}.$$

and we have $g'(x)\equiv 0$. Thus it must be $g''(x)\equiv 0$ and $g(x)=Dx+E$.

Equality (11) can be written now as

$$2A = a_{ij}(2Ax_i+B)D+a_{ji}(2Ax_j+B)D \tag{12}$$

whence we obtain for $x_i=x_j=0$

$$2A = (a_{ij}+a_{ji})BD. \tag{13}$$

Differentiating (12) for $x_i$ and $x_j$ $(i\neq j)$, we obtain $a_{ij}AD=0$ and $a_{ji}AD=0$. It must be $AD=0$, since in the contrary case we would have $\varphi(x)\equiv 0$. We shall show that $A=0$. In fact, if $A\neq 0$, then $D=0$, but this contradicts (13). Thus the function $\varphi$ has the form

$$\varphi(x) = Bx+C.$$

Now, one can write (12) as $(a_{ij}+a_{ji})BD=0$. It must be $BD=0$, since $BD\neq 0$ yields $\varphi(x)\equiv 0$.

If $B=0$, then $\varphi(x)=C$ $(C\neq 0)$ and we conclude from equation (2) that

$$g(x) = \frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}}.$$

If $D=0$, then $g(x)=E$ and it results from (2) that

$$E = \frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}}, \qquad \text{for} \quad C\neq 0$$

and

$$E = \frac{n}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}} = \frac{1}{\sum\limits_{\substack{j=1\\ i\neq j}}^{n} a_{ij}} \qquad (i=1, \ldots, n) \quad \text{for} \quad C=0.$$

Thus the solutions of equation (2) have either form 1° or 2°.

## References


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