

On minimal ideals in the circle composition semigroup of a ring

By. W. EDWIN CLARK (Gainesville, Fla.) and Jacques Lewin (Syracuse, NY.)

Let R be a ring, and let $\circ$ denote the circle composition on R defined by $a \circ b = a + b - ab$. In this note we discuss the existence of minimal ideals in $(R, \circ)$ and their significance in R . In § 1 we show that $(R, \circ)$ has a completely simple minimal ideal K if and only if R contains an idempotent e which is an identity for R modulo its Jacobson radical. If this is the case then the ideal $I(R)$ of the ring R generated by $K - K$ is seen to be a radical-like ideal which is zero if and only if R has an identity. A necessary and sufficient condition is given for R to be a splitting extension of $I(R)$ by a subring eRe for an idempotent e of R .

An interesting question is the determination of those simple rings R for which $(R, \circ)$ is simple. Šašić's example ([5]) shows that R may be a radical ring. Our results imply that if $(R, \circ)$ is completely simple then R must be a radical ring. Example B below shows, however, that there are semi-simple non-simple rings for which $(R, \circ)$ is simple. By taking quotients one may easily construct from the ring in example B a ring R for which both $(R, \circ)$ and the multiplicative semigroup $(R, \cdot)$ are simple.

1. We denote by $J(R)$ the (Jacobson) radical of the ring R (see [4], ch. 1).

An idempotent e in R will be called *principal* if

$$(1 - e)R + R(1 - e) \subset J(R).$$

(Note that we use 1 only as a notational device. R may or may not have an identity). A principal idempotent is always principal in the classical sense (see [1], p. 25); however the converse does not hold. If $e = 0$ is principal, then $J(R) = R$. If R has an identity 1, then 1 is the only principal idempotent of R . Our results become trivial in both these cases.

It is easy to see that a semi-primary SBI ring ([4], ch. III), and hence a ring with d.c.c., has a principal idempotent.

Let e be an idempotent. We define

$$P_e = (1 - e)Re + eR(1 - e) + (1 - e)R(1 - e).$$

P_e is the sum of the last three terms of the two-sided Peirce decomposition of R with respect to e . We note that $R = eRe + P_e$ is a direct sum decomposition qua abelian groups and that

$$P_e = (1 - e)R + R(1 - e).$$

Thus an idempotent e is principal if and only if $P_e \subset J(R)$.

We further define

$$G_e = e \circ J(R) \circ e = (1-e)J(R)(1-e) + e$$

$$L_e = J(R) \circ e = J(R)(1-e) + e$$

$$R_e = e \circ J(R) = (1-e)J(R) + e$$

$$K_e = L_e \circ R_e = J(R) \circ e \circ J(R).$$

Lemma. 1. $(G_e, \circ)$ is a group for any idempotent e in R .

PROOF. Since $e = e \circ e = e \circ 0 \circ e$, it follows that $e \in G_e$ and that e is an identity for G_e . Let now $g \in G_e$. Then $g = y + e$ where $y \in (1-e)J(R)(1-e) \subseteq J(R)$. Since $y \in J(R)$, y has a quasi-inverse z . Since $0 = y \circ z = y + z - yz$, it follows that $z = yz - y = (1-e)yz - (1-e)y = (1-e)(yz - y) = (1-e)z$. Similarly $z = z(1-e)$. Hence $z \in (1-e)J(R)(1-e)$ and $h = z + e$ is in G_e . Now, $ez = ze = ey = ye = 0$ and $z \circ y = y \circ z = 0$. Thus $h \circ g = g \circ h = e$, every element of G_e has an inverse, and the lemma is proved.

A semigroup is called *simple* if it contains no proper ideals, and *completely simple* if it is simple and every element lies in a subgroup, i.e. a subsemigroup which is a group (see [3]). We shall need the following result:

Theorem. (A. H. CLIFFORD [2]). *If L and R are minimal left and minimal right ideals respectively of a semigroup S , then $K = LR$ is a minimal (two-sided) ideal of S and is a completely simple semigroup.*

This enables us to prove

Lemma 2. *If e is a principal idempotent of R , then K_e is a completely simple minimal ideal of $(R, \circ)$.*

PROOF. Since $K_e = L_e \circ R_e$, it suffices, by Clifford's Theorem, to show that L_e and R_e are minimal left and right ideals. By symmetry it suffices to show that $L = L_e$ is a minimal left ideal.

Using the fact that e is principal, and thus that $R(1-e) = J(R)(1-e)$ we have that

$$R \circ L = R \circ J(R) \circ e \subset R \circ e = R(1-e) + e = J(R)(1-e) + e = L.$$

Thus L is a left ideal. Let now T be any left ideal of $(R, \circ)$ with $T \subset L$. Then $e \circ T \subset L \cap T \cap G_e$ since $e \circ L = G_e$. Therefore G_e meets T and, since G_e is a group, $G_e \subset T$. In particular, $e \in T$ and $L = J(R) \circ e \subset T$. Thus $T = L$ and L is indeed minimal.

Corollary 3. If e and f are any two principal idempotents, then $K_e = K_f$.

Corollary 4. If e is a principal idempotent, then

$$G_e = e \circ R \circ e$$

$$L_e = R \circ e$$

$$R_e = e \circ R$$

$$K_e = R \circ e \circ R.$$

PROOF. Since $e \in L_e$, $R \circ e \subset L_e$. But L_e is minimal. Thus $R \circ e = L_e$. The other parts follow similarly.

Theorem 1. *If R is a ring, $(R, \circ)$ has a completely simple minimal ideal K if and only if R contains a principal idempotent. Further, an idempotent is in K if and only if it is principal.*

PROOF. The sufficiency was proved in Lemma 2 and Corollary 3. To show the necessity of the conditions, assume that K is a completely simple minimal ideal of $(R, \circ)$ and let e be any idempotent of K . We show that e is principal. First we note that $(1-e)R = e \circ R - e$. Let then $x = e \circ r - e$ for some $r \in R$. Since K is completely simple $e \circ r \circ e$ lies in the subgroup $e \circ K \circ e = e \circ R \circ e$ of K containing e (see [3], Lemma 2.46, p. 77). There then exists $s \in R$ with

$$(1) \quad e = (e \circ s \circ e) \circ (e \circ r \circ e) = e \circ (s \circ e \circ r) \circ e.$$

Let $z = e \circ s \circ e$ and $y = zx - x$. This yields $zx = (1-e)s(1-e)r$. Thus $y \in (1-e)R$. If we can show that $y \circ x = 0$, then it will follow that $(1-e)R$ is a quasi regular right ideal and therefore is contained in $J(R)$, ([4], ch. 1). We now calculate:

$$\begin{aligned} y \circ x &= zx - zx^2 + x^2 \\ &= (1-e)(s - sr + ser + r)(1-e)r \\ &= (1-e)(s + e - se + r - sr - er + ser)(1-e)r \\ &= (1-e)(s \circ e \circ r)(1-e)r. \end{aligned}$$

Now, by (1) we have

$$e = e \circ (s \circ e \circ r) \circ e = (1-e)(s \circ e \circ r)(1-e) + e$$

which implies that $(1-e)(s \circ e \circ r)(1-e) = 0$ and $y \circ x = 0$. Similarly, $R(1-e) \subset J(R)$. Thus e is principal. Since e was any idempotent of K , the theorem is proved.

2. In this section, R is a ring with a principal idempotent e and K is the minimal ideal of $(R, \circ)$.

We denote by $I(R)$ the ideal of R generated by $K - e$. $I(R)$ does not depend on the choice of e since $K - a$ and $K - b$ generate the same ideal whenever a and b are both elements of K . Since $K \subset J(R) + e$, it follows immediately that $I(R) \subset J(R)$.

Theorem 2. *If R contains a principal idempotent, then*

- (i) $I(R) = 0$ if and only if R has an identity
and
(ii) $I(R/I(R)) = 0$.

PROOF. (i) If $I(R) = 0$, then $K - e = 0$. Since $R \circ e \subset K$ we must also have that $R \circ e - e = 0$. This implies that e is a right identity for R . Likewise, e is a left identity. Conversely, if e is an identity for R , then $K = \{e\}$ and $I(R) = 0$. The verification of (ii) is routine and will be omitted.

Since $I(R)\phi \subset I(R\phi)$ for any homomorphism ϕ of R , we easily obtain

Corollary 5. $I(R)$ is the intersection of all ideals A of R for which R/A has an identity.

We can now connect $I(R)$ and the Peirce decomposition of R :

Lemma 6. $I(R)$ is the subring generated by P_e .

PROOF. Since $R = eRe + P_e$, the factor by the ideal generated by P_e has an identity and thus this ideal contains $I(R)$. However, $(1-e)R \cup R(1-e) \subset (L_e - e) \cup (R_e - e) \subset K - e$. Since $P_e = (1-e)R + R(1-e)$ this implies that $P_e \subset I(R)$. To complete the proof of the lemma, we need only observe that the subring and the ideal generated by P_e coincide.

Corollary 7. If either $I(R)^2 = 0$ or R contains a principal central idempotent, then $R = eRe + I(R)$ and $eRe \cap I(R) = 0$.

PROOF. One need only notice that in both these cases P_e is already a subring, and thus that $I(R) = P_e$.

By a *linear variety* in a ring R we mean a translate $a + M$ of a subgroup M of the additive group of R . It is easily verified that V is a linear variety of R if and only if all finite sums $\sum n_i v_i$ lie in V whenever the $v_i \in V$ and the n_i are integers whose sum is 1.

It is easy to verify that if T is an ideal of $(R, \circ)$ the linear variety generated by T is also an ideal of $(R, \circ)$ and that if T is both an ideal of $(R, \circ)$ and a linear variety then, for any $t \in T$, $T - t$ is an ideal of R . However all the ideals of R may not be obtainable in this fashion.

Theorem 3. Let e be a principal idempotent of R . Then, K is a linear variety of R if and only if $R = eRe + I(R)$ (a direct sum qua abelian groups). When this occurs, then $K = P_e + e$ and $I(R) = P_e$.

PROOF. We first assume that K is a linear variety. We need only show that $I(R) = P_e$. To this effect, we first show that $eKe = e$: Let $x \in K$. Since K is a linear variety, as well as an ideal of $(R, \circ)$ containing e , we have that

$$exe = x - x \circ e - e \circ x + e \circ x \circ e + e$$

lies in K . Since K is a union of groups by Theorem 1, there exists an idempotent $g \in K$ for which $g \circ exe = exe = exe \circ g$. Thus $g = gexe$ and $g = exeg$ and further $ge = eg = g$. Whence $e \circ g = e = g \circ e$. Therefore $e = g \circ e \circ g \in g \circ R \circ g$ which is a group by lemma 1 and corollary 4. Since a group contains a single idempotent, e must equal g . This shows that $exe = e$ for all $x \in K$.

Now, by a previous remark, since K is a linear variety, $K - e$ is an ideal of R and therefore $K - e = I(R)$. Thus $eI(R)e = e(K - e)e = eKe - e = e - e = 0$ and it follows easily that $I(R) \subset P_e$. Lemma 6 then forces P_e to coincide with $I(R)$.

Assume now, to prove the converse, that $R = eRe + I(R)$ is a direct sum qua abelian groups. Since $R = eRe + P_e$ is such a direct sum and $P_e \subseteq I(R)$, it follows that $P_e = I(R)$. Recalling that $K - e$ generates $I(R) = P_e$, we see that $K - e \subseteq P_e$. To prove the opposite inclusion we show first that any idempotent $y + e \in P_e + e$ must lie in K :

$(e + y)^2 = e + y$ and hence $ey + ye + y^2 = y$. Multiplying by e on the left yields $eye + ey^2 = 0$ and hence $(e + ey) \circ (e + y) = (e + y)$. To show that $e + y \in K$, it therefore suffices to show that $e + ey \in K$. However this is clear since $e + eP_e = e + eR(1 - e) \subset L_e \subset K$.

Let now $e+a$ be an arbitrary element of $e+P_e$. Since $a \in P_e = I(R)$ we have that $a-2ea \in P_e$. Let q be the quasi-universe of $a-2ea$ in P_e . (Such a q exists since P_e is an ideal contained in $J(R)$). Let $y = eq + qe + qeq$. Since $eP_e e = 0$, it follows that $y+e$ is an idempotent in P_e+e and thus that $y+e \in K$. If we can show that $(y+e) \circ (a+e) = a+e$ we will have shown that $a+e \in K$ and thus that $K = P_e + e$, which will prove that K is a linear variety. Now,

$$(y+e) \circ (a+e) = a + e + y - ya - ye - ea.$$

Substituting $eq + qe + qeq$ for y and using the fact that $eqe = 0$, we find that

$$(y+e) \circ (a+e) = a + e + (1+q)(eq - ea - eqa).$$

It therefore suffices to show that $eq - ea - eqa = 0$. However,

$$0 = e0 = e(q \circ (a - 2ea)) = e(q + a - 2ea - qa + 2qea) = eq - ea - eqa.$$

This then completes the proof of Theorem 3.

3. Examples.

A. The ring of all real 3×3 matrices of the form

$$\begin{pmatrix} x & 0 & 0 \\ y & 0 & 0 \\ z & r & s \end{pmatrix}$$

provides an example of a ring R where K is not a linear variety. This is easily verified since K consists of all matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ -xy & y & 0 \end{pmatrix}$$

B. We now give an example of a ring R for which $K = R$ is simple but not completely simple:

Let L be the ring of all linear transformations on a vector space V of uncountable dimension, and let R be the subring of L consisting of all linear transformations σ such that $\dim V\sigma < \dim V$.

We first note that any ideal I of $(R, \circ)$ must contain an idempotent, for if $\sigma \in I$ and e is any projection on $V\sigma$, then $e = \sigma \circ e \in I$. To prove that $(R, \circ)$ is simple we need only show that any ideal of $(R, \circ)$ contains 0, and we achieve this by exhibiting, for any idempotent $e \in R$, σ and τ in R for which $\sigma \circ e \circ \tau = 0$. Let then $e \in R$ be the projection on the subspace M_1 along the subspace W . Since $\dim W > \dim M_1$, we can find a sequence of subspaces $M_2, M_3, M_4, \dots$, all isomorphic to M_1 such that

$$W = (M_2 \oplus M_3 \oplus \dots) \oplus U.$$

We choose bases $\{m_\alpha^i : i = 1, 2, \dots; \alpha \in A\}$ for the M_i and we define σ and τ by

$$\begin{aligned} m_\alpha^1 \sigma &= m_\alpha^2; & m_\alpha^i \sigma &= m_\alpha^i + m_\alpha^{i+1} & i > 1, & U\sigma &= 0 \\ m_\alpha^1 \tau &= m_\alpha^1; & m_\alpha^i \tau &= m_\alpha^i + m_\alpha^{i-1} & i > 1, & U\tau &= 0 \end{aligned}$$

Then it is clear that both σ and τ are in R , and one verifies that $\sigma \circ e \circ \tau = 0$. $(R, \circ)$ is then simple.

If $(R, \circ)$ were completely simple, then, by Theorem 1, all idempotents of R would be principal. But it is clear that R has no principal idempotents.

References

- [1] A. A. ALBERT, Structure of Algebras, *Amer. Math. Soc. Colloquium Publication*, **24** (1939).
- [2] A. H. CLIFFORD, Semigroups containing minimal ideals *Amer. J. Math.* **70** (1948), 521—526.
- [3] A. H. CLIFFORD and G. B. PRESTON, The algebraic theory of semigroups, Vol. I, *Math. Surveys*, No. 7 *Amer. Math. Soc.*, 1961.
- [4] N. JACOBSON, Structure of Rings *Amer. Math. Soc. Colloquium Publication*, **37** (1956).
- [5] E. SAŚIADA, Solution of the problem of the existence of a simple radical ring, *Bull. Acad. Polon. Sci.*, **9** (1961) p. 257.

(Received April 29, 1966.)

The authors would, respectively, like to extend their thanks to the Ford Foundation and the Office of Naval Research who supported this work.