

Polyadic semigroups

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1. Introduction

A *polyadic semigroup*, specifically an *n-semigroup*, is an associative *n*-ary operation F on a set $\mathfrak{M}$. I. e., F is a function whose domain is the Cartesian power $\mathfrak{M}^n$ and whose range is a subset of $\mathfrak{M}$, such that each of the $n-1$ equations ¹⁾

$$(1) \quad FFx_1 \dots x_{2n-1} = Fx_1 \dots x_k Fx_{k+1} \dots x_{2n-1} \quad (1 \leq k \leq n-1)$$

holds for every $(2n-1)$ -tuple $x_1 \dots x_{2n-1}$ of elements of $\mathfrak{M}$. The main purpose of this note is to prove the following result:

Theorem 1. Let F be an *n-semigroup* on a set $\mathfrak{M}$. Assume that F has a *right identity*, i. e., there exists an $(n-1)$ -tuple $b_1 b_2 \dots b_{n-2} a$ (briefly, αa) such that:

$$(3) \quad Fxb_1 \dots b_{n-2} a = Fx\alpha a = x$$

for every x in $\mathfrak{M}$. Define a binary operation S_α , and a unary operation f by:

$$(3) \quad S_\alpha xy = Fx\alpha y,$$

$$(4) \quad fx = Fax\alpha,$$

for all x, y in $\mathfrak{M}$. Let $c = Fa^n$. Then S_α is a (binary) semigroup with right-identity a , f is an endomorphism of S_α , and we have

$$(5) \quad Fx_1 x_2 \dots x_n = S_\alpha^n x_1 f x_2 \dots f^{n-1} x_n c$$

for all n -tuples $x_1 x_2 \dots x_n$ in $\mathfrak{M}^n$.

This theorem, which will be proved in § 2, is a generalization of a theorem of M. Hosszú ([2]) for polyadic groups. A polyadic group (*n-group*) ²⁾ is an *n-semigroup* in which, given any n of the $n+1$ elements $x_1, \dots, x_{n+1}$ of $\mathfrak{M}$, a solution of the equation

$$(6) \quad Fx_1 \dots x_n = x_{n+1}$$

¹⁾ I use the parentheses-free (Lukasiewicz) functional notation. Expressions such as FF , FFF , aa , $aaaa$ will be abbreviated to F^2 , F^3 , a^2 , a^4 , etc.

²⁾ This definition, as well as that of generalized associativity (1), was first introduced by E. Dörnte [1].

for the remaining unknown element of $\mathfrak{M}$ always exists. In other words, each one-place part function ³⁾ of F is onto $\mathfrak{M}$. This phrasing corresponds to the Huntington definition of a (binary) group as an associative system G where $aG = Ga = G$ ([3]). As in the binary case, it then can be shown that these 1-place part-functions are one-to-one, i.e., the solutions of (6) are unique.

E. L. POST ([4]) in his fundamental Coset Theorem showed that every polyadic group F on a set $\mathfrak{M}$ has a covering (binary) group K . This means that there exists a (unique) group K generated by $\mathfrak{M}$, such that for any n -tuple $x_1 \dots x_n$ of elements of $\mathfrak{M}$ we have:

$$(7) \quad Fx_1 \dots x_n = K^{n-1}x_1 \dots x_n,$$

and such that $\mathfrak{M}$ is a generating coset of a cyclic factor group of K .

The use of the covering group K has the disadvantage of introducing elements not in $\mathfrak{M}$. In [2], Hosszú showed that such introduction can be avoided, and that any polyadic group can be realized as a composite of a binary group G on $\mathfrak{M}$ and an automorphism f of G . Specifically, he proved that

$$(8) \quad Fx_1 \dots x_n = G^n x_1 f x_2 \dots f^{n-2} x_n c,$$

where c is a fixed element of $\mathfrak{M}$, $fc = c$ and $f^{n-1}x = C^2 c x c^{-1}$, i. e., f^{n-1} is an inner automorphism of G .⁴⁾

In § 3, we show that Hosszú's theorem is a special case of Theorem 1. Finally, in § 4 we use Theorem 1 to obtain a generalization to n -semigroups of a theorem of Dörnte on derivability of n -groups.

2. Proof of Theorem 1

Lemma 1. *The function S_α defined in (3) is a (binary) semigroup, i.e.,*

$$(9) \quad S_\alpha S_\alpha xyz = S_\alpha x S_\alpha yz,$$

for all x, y, z in $\mathfrak{M}$.

$$\text{PROOF.} \quad SSxyz = FSxy\alpha z = (3)$$

$$= FFx\alpha y\alpha z = (3)$$

$$= Fx\alpha Fy\alpha z = (1)$$

$$= Fx\alpha Syz = (3)$$

$$= SxSyz. \quad (3)$$

Note that in this proof we did not have to assume the existence of any identity for F . If F has a right-identity αa , then it is immediate from (3) that a is a right-identity for S_α .

³⁾ A k -place part function of F is derived from F by fixing $n-k$ of the arguments.

⁴⁾ As a byproduct, Hosszú was able to immediately obtain a theorem of E. VINCZE [5], which itself generalized a theorem of J. ACZÉL on real solutions of the functional equation of associativity.

Lemma 2. The function f defined in (4) is an endomorphism of S_α , i.e.,

$$(10) \quad fS_\alpha xy = S_\alpha fxfy,$$

for all x, y in $\mathfrak{M}$.

$$\text{PROOF. } S_\alpha fxfy = S_\alpha Fax\alpha Fay\alpha = \quad (4)$$

$$= FFax\alpha Fay\alpha = \quad (3)$$

$$= FFaxb_1 \dots b_{n-2}\alpha Fay\alpha =$$

$$= FaFx \dots Fb_{n-2}\alpha ay\alpha = \quad (1)$$

$$= FaFxb_1 \dots b_{n-2}y\alpha = \quad (2)$$

$$= FaFx\alpha y\alpha =$$

$$= fFx\alpha y = \quad (4)$$

$$= fS_\alpha xy. \quad (3)$$

Lemma 3. For the functions f and S_α of Theorem 1, we have:

$$(11) \quad S_\alpha^n x_1 f x_2 \dots f^{n-1} x_n c = S_\alpha^2 x_1 f S_\alpha x_2 \dots f S_\alpha x_{n-1} f x_n c.$$

PROOF. The proof depends on the fact that if f is an endomorphism of S_α , then so are all the iterates of f . Hence for any positive integer m , we have

$$(4a) \quad f^m S_\alpha xy = S_\alpha f^m x f^m y.$$

Now starting with the left hand side of (11), we have:

$$S_\alpha^n x_1 f x_2 \dots f^{n-1} x_n c = S_\alpha^2 x_1 S_\alpha f x_2 \dots S_\alpha f^{n-2} x_{n-1} f^{n-1} x_n c = \quad (9)$$

$$= S_\alpha^2 x_1 S_\alpha f x_2 \dots S_\alpha f^{n-2} x_{n-1} f^{n-2} f x_n c =$$

$$= S_\alpha^2 x_1 S_\alpha f x_2 \dots f^{n-2} S_\alpha x_{n-1} f x_n c \quad (4a)$$

Repeating this procedure leads us, after $(n-2)$ more steps, to (11).

Remark: Although f is an endomorphism of S_α , fS_α is in general *not*⁵⁾ a semigroup on $\mathfrak{M}$. For example, let $\mathfrak{M}$ be the set of real numbers, and S ordinary real addition. For any $a \neq 0, 1$ let f_a denote the linear function defined by $f_a x = a \cdot x$. Then f_a is an endomorphism of S . But we have:

$$f_a S f_a S x y z = a(a(x+y)+z) = a^2 x + a^2 y + a z \neq$$

$$\neq a x + a^2 y + a^2 z = (a(x+a(y+z))) = f_a S x f_a S y z.$$

A sufficient condition for fS to be associative is that f be idempotent; in this case, if f is onto $\mathfrak{M}$, then f is the identity function on $\mathfrak{M}$.

⁵⁾ Occasionally one finds the statement such as: If f is an endomorphism of the semigroup S then the image fS is a (sub-) semigroup. This is the result of using the same symbol „ S ” to denote the set $\mathfrak{M}$ and the semigroup, i.e., the pair $(\mathfrak{M}, S)$. What is meant is: The pair $(f\mathfrak{M}, S')$, where S' is the restriction of S to the set $f\mathfrak{M}$, is a subsemigroup of $(\mathfrak{M}, S)$.

Lemma 4. *If $c = Fa^n$, then*

$$(12) \quad Fx_1 \dots x_n = S_x^2 x_1 f S_x x_2 \dots f S_x x_{n-1} f x_n c.$$

PROOF.

$$Fx_1 \dots x_n = F F x_1 \alpha a F x_2 \alpha a \dots F x_{n-1} \alpha a F x_n \alpha a = \quad (2)$$

$$= F F x_1 \alpha F a x_2 \alpha \dots F a x_{n-1} \alpha F a x_n \alpha a = \quad (1)$$

$$= F F x_1 \alpha F a x_2 \alpha \dots F a x_{n-1} \alpha f x_n a = \quad (4)$$

$$= F F x_1 \alpha F a x_2 \alpha \dots F a x_{n-1} \alpha F f x_n \alpha a a = \quad (2)$$

$$= F F x_1 \alpha F a x_2 \alpha \dots F a F x_{n-1} \alpha f x_n \alpha a^2 = \quad (1)$$

$$= F F x_1 \alpha F a x_2 \alpha \dots F a S_x x_{n-1} f x_n \alpha a^2 = \quad (3)$$

$$= F F x_1 \alpha F a x_2 \alpha \dots f S_x x_{n-1} f x_n a^2. \quad (4)$$

Applying the procedure in the last 4 steps $n-2$ times in succession we obtain:

$$Fx_1 \dots x_n = F F x_1 \alpha f S_x x_2 \dots f S_x x_{n-1} f x_n a^{n-1} = \quad (2)$$

$$= F F x_1 \alpha F f S_x x_2 \dots f S_x x_{n-1} f x_n \alpha a^{n-1} = \quad (2)$$

$$= F F x_1 \alpha f S_x x_2 \dots f S_x x_{n-1} f x_n \alpha F a^n = \quad (1)$$

$$= F S_x x_1 f S_x x_2 \dots f S_x x_{n-1} f x_n \alpha c = \quad (2)$$

$$= S_x^2 x_1 f S_x x_2 \dots f S_x x_{n-1} f x_n c. \quad (2)$$

Equations (11) and (12) together yield (5), and so the proof of Theorem 1 is complete.

N. B. Similar arguments will show that a semigroup with a left identity αx can be realized in the form:

$$Fx_1 \dots x_n = S_x^n c g^{n-1} x \dots g x_{n-1} x_n,$$

where $S_x xy = Fx\alpha y$, $gx = F\alpha xa$, and $c = Fa^n$.

3. Hosszú's Theorem

Lemma 5. *If F is an n -group, α is any fixed $(n-2)$ tuple, and S_α is defined via (3), then S_α is a group.*

PROOF. By Lemma 1, S_α is a semigroup. And since F is an n -group, any equation of the form

$$S_\alpha x_1 x_2 = x_3$$

has a solution whenever two of the x 's are given. But this is one of the standard definitions of a group.

In dealing with groups, we shall generally write G_α instead of S_α .

Lemma 6. *For every $(n-2)$ -tuple $\alpha \in \mathfrak{M}^{n-2}$, there exists an $a \in \mathfrak{M}$ such that αa and $a\alpha$ are each (two-sided) identities of the n -group F .*

PROOF. The (binary) group G_α has an identity, say a . Then $G_\alpha ax = Fa\alpha x = x = G_\alpha x a = Fx\alpha a$. Thus αa is a right identity and $a\alpha$ a left identity of F . Now, to show that $a\alpha$ is also a right identity we have:

$$\begin{aligned} Fx\alpha a &= Fxab_1 \dots b_{n-2} = Fxab_1 \dots Fb_{n-2}\alpha a = FxFab_1 \dots b_{n-2}\alpha a = \\ &= FxFa\alpha b_1 \dots b_{n-2}a = Fxb_1 \dots b_{n-2}a = Fx\alpha a = x. \end{aligned}$$

Similarly, we can show, that αa is also a left identity.

This lemma is a special case of a more general theorem of POST ([4]): Given any $n-2$ elements $a_1 \dots a_{i-1} a_{i+1} \dots a_{n-1}$ of $\mathfrak{M}$ ($i=1, 2, \dots, n-1$), there exists an element a_i in $\mathfrak{M}$ such that $a_1 \dots a_{n-1}$ is an identity of the n -group; furthermore, any cyclic permutation of $a_1 \dots a_{n-1}$ is also an identity.

Lemma 7. Let $a\alpha$ be a right identity of F and $c = Fa^n$. If f is as defined in (4), then c is a fixed point of f .

$$\text{PROOF. } fc = Fac\alpha = FaFa^n\alpha = FFa^n\alpha = FFa^n\alpha\alpha = Fca\alpha = c.$$

Note that the proof does not require F to be an n -group.

Lemma 8. If F is an n -group, then f is an automorphism of G_α and f^{n-1} an inner automorphism of G_α : Specifically $f^{n-1}x = G_\alpha^2 cxc^{-1}$ where c is as in Lemma 7.

PROOF. 1. By Lemma 2, f is an endomorphism of G_α . Since F is an n -group, f is one-to-one onto $\mathfrak{M}$, i. e., an automorphism.

$$\begin{aligned} 2. f^{n-1}x &= F^{n-1}a^{n-1}x\alpha^{n-1} = F^{n-1}a^{n-2}a\alpha^{n-1} = F^{n-1}a^{n-2}Fa\alpha x\alpha^{n-1} = \\ &= FFFa^n\alpha x\alpha F^{n-3}\alpha^{n-2} = FFC\alpha x\alpha F^{n-3}\alpha^{n-2} = FG_\alpha c\alpha F^{n-3}\alpha^{n-2} = G_\alpha^2 c\alpha F^{n-3}\alpha^{n-2} \end{aligned}$$

To complete the proof we need only show that $F^{n-3}\alpha^{n-2} = c^{-1}$, i.e., that $G_\alpha cF^{n-3}\alpha^{n-2} = a$. Now

$$\begin{aligned} GcF^{n-3}\alpha^{n-2} &= GFa^nF^{n-3}\alpha^{n-2} = FFa^n\alpha F^{n-3}\alpha^{n-2} = \\ &= F^{n-1}a^n\alpha^{n-1} = F^{n-2}a^{n-1}Fa\alpha b_1 \dots b_{n-2}\alpha^{n-3}, \end{aligned}$$

where $b_1 \dots b_{n-2} = \alpha$. Since $a\alpha$ is a left identity, we have

$$GcF^{n-3}\alpha^{n-2} = F^{n-2}a^{n-1}b_1 \dots b_{n-2}\alpha^{n-3} = F^{n-2}a^{n-1}\alpha^{n-2}.$$

Repeating the procedure, we reduce the right hand side to $Fa\alpha a = a$.

Lemmas 5, 6, 7 and 8 together show that Hosszú's Theorem is a special case of Theorem 1.

4. Derivable n -semigroups

If $Fa^n = a$, we call a an idempotent element of F .

Lemma 9. Let F be an n -semigroup with an idempotent element a , and let a function f be defined by

$$(13) \quad fx = Fxa^{n-2}.$$

Then $f^{n-1}x = F^2a^{n-1}xa^{n-1}$, $f^n = f$ and f^{n-1} is an idempotent function, i.e., $f^{n-1}f^{n-1} = f^{n-1}$.

$$\begin{aligned} \text{PROOF 1. } f^{n-1}x &= F^{n-1}a^{n-1}xa^{(n-2)(n-1)} = F^{n-1}a^{n-1}xa^n a^{(n-2)(n-1)-n} = \\ &= F^{n-2}a^{n-1}x F a^n a^{(n-2)(n-1)-n} = F^{n-2}a^{n-1}x a a^{(n-2)(n-1)-n} = \\ &= F^{n-2}a^{n-1}xa^{(n-3)(n-1)}. \end{aligned}$$

Repeating the above steps $n-2$ more times we get:

$$f^{n-1}x = F^2a^{n-1}xa^{n-1}.$$

$$\begin{aligned} 2. \quad f^n x &= f^{n-1}fx = F^2a^{n-1}fx a^{n-1} = F^2a^{n-1}Fa x a^{n-2}a^{n-1} = \\ &= F^2a^n x F a^n a^{n-3} = Fa x a a^{n-3} = Fa x a^{n-2} = fx. \\ 3. \quad f^{n-1}f^{n-1} &= f^{2n-2} = f^n f^{n-2} = ff^{n-2} = f^{n-1}. \end{aligned}$$

Lemma 10. Let F be an n -semigroup on a set $\mathfrak{M}$. Let a be an element of $\mathfrak{M}$, and let f be as defined in (13). Then the following statements are equivalent: (i) a is an idempotent element of F , and f is onto $\mathfrak{M}$; (ii) a^{n-1} is a (2-sided) identity of F .

PROOF. (i) implies (ii).

Since f is onto $\mathfrak{M}$, f^{n-1} is onto $\mathfrak{M}$. But, by Lemma 9, f^{n-1} is idempotent, and an idempotent function onto a set is the identity function. We therefore have:

$$\begin{aligned} x &= f^{n-1}x = F^2a^{n-1}xa^{n-1} = F^2aa^{n-2}xa^{n-1} = F^2Fa^n a^{n-2}xa^{n-1} = \\ &= F^2a^{n-1}(Fa^{n-1}x)a^{n-1} = f^{n-1}Fa^{n-1}x = Fa^{n-1}x, \end{aligned}$$

i.e., a^{n-1} is a left identity of F . Similarly we can show that a^{n-1} is a right identity. (ii) implies (i).

$$1. \quad Fa^n = Faa^{n-1} = a \text{ i.e., } a \text{ is idempotent.}$$

$$2. \quad f^{n-1}x = F^2a^{n-1}xa^{n-1} = Fa^{n-1}Fx a^{n-1} = Fa^{n-1}x = x,$$

i.e., f^{n-1} is onto $\mathfrak{M}$. Since the range of f^{n-1} is a subset of the range of f , the proof is complete.

DÖRNTE ([1]) already distinguished between genuine (echte) and derivable (ableitbare) n -groups. An n -semigroup F is derivable from a binary semigroup S if $F = S^{n-1}$.

Theorem 2. Let F be an n -semigroup on a set $\mathfrak{M}$. Then the following statements are equivalent: (i) F is derivable from a (binary) semigroup S on $\mathfrak{M}$ with identity a ; (ii) F has an idempotent element a such that $Faxa^{n-2} = x$ for all $x \in \mathfrak{M}$; (iii) there is an element $a \in \mathfrak{M}$ such that a^{n-1} is a right identity of F , and such that $Faxa^{n-2} = x$ for all $x \in \mathfrak{M}$.

PROOF. (i) implies (ii). $Fa^n = S^{n-1}a^n = a$, i.e., a is an idempotent element of F . Furthermore

$$Fa x a^{n-2} = S^{n-1}a x a^{n-2} = x.$$

(ii) implies (iii). Since $fx = Fxa^{n-2} = x$ means that f is onto $\mathfrak{M}$, this implication follows from Lemma 10.

(iii) implies (i). Let $\alpha = a^{n-2}$. By hypothesis αa is a (right) identity. We can therefore apply Theorem 1 and obtain

$$Fx_1 \dots x_n = S_\alpha^n x_1 f x_2 \dots f^{n-1} x_n F a^n = S_\alpha^n x_1 f x_2 \dots f^{n-1} x_n a.$$

Now $fx = Fxa = Fxa^{n-2} = x$ by hypothesis, and a is the identity of S_α . Hence we have

$$Fx_1 \dots x_n = S_\alpha^{n-1} x_1 \dots x_n.$$

Since any group has an identity, Theorem 2 includes as a special case the necessary and sufficient criterion for n -groups to be derivable from binary groups (Dörnte, [1], § 2, Theorem 6).

Acknowledgment

I wish to acknowledge my thanks to Dr. A. SKLAR for his many helpful discussions and criticisms during the preparation of this paper.

Added in proof. I am indebted to Professor B. M. SCHEIN for directing my attention to the fundamental work of L. M. GLUSKIN on n -semigroups and related systems (Positional Operatives, Doklady Akad. Nauk SSSR **157**, 1964, 767—770; Positional Operatives, Mat. Sbornik **68**, 1965, 444—472). In particular, Theorem 5 in the Doklady paper is essentially a weaker form of Theorem 1 of this paper.

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(Received August 18, 1966.)