On the zeros of solutions of ordinary second order differential equations

By Á. ELBERT (Budapest)

We consider the solution of the differential equation

$$y'' + q(x)y = 0, \tag{1}$$

where $q(x)$, $q'(x)$, $q''(x)$ are continuous functions, $q(x) > 0$, $q'(x) > 0$ for $x > 0$, and

$$\alpha q'^2(x) - q(x)q''(x) \geqq 0 \quad \text{for} \quad x > 0, \tag{2}$$

with the initial conditions $y(0) = 0$, $y'(0) \neq 0$. Using the notation $\sigma = \sigma(x) = \sqrt{q(x)}$ we obtain from (2)

$$(2\alpha - 1)\sigma'^2(x) - \sigma(x)\sigma''(x) \geqq 0 \quad \text{for} \quad x > 0, \tag{3}$$

where we put in the following $\beta = 2\alpha - 1$.

We denote the first n roots of the equation $y(x) = 0$, if they exist, by $x_0 = 0, x_1, \ldots, x_n$, those of $y'(x) = 0$ by $x_1', x_2', \ldots, x_n'$, where

$$x_0 = 0 < x_1' < x_1 < \ldots < x_n' < x_n.$$

For the root x_i it follows from a result of E. MAKAI (see [1]), that the inequality

$$\int_0^{x_i} \sigma(x)\, dx < i\pi \qquad (i = 1, 2, \ldots, n) \tag{4}$$

holds, if α is not greater than $\frac{5}{4}$.

Makai conjectured (see [2]) that the quantities on the left of (4) are in certain circumstances greater than $(i - \frac{1}{2})\pi$. In this paper we will prove the following

Theorem. *If the inequality $\alpha < 1$ holds for the quantity α in* (2) *then the inequalities*

$$\begin{cases} \displaystyle\int_0^{x_i} \sigma(x)\, dx > i\pi - \frac{1}{3 - 2\alpha}\frac{\pi}{2} & (i = 1, 2, \ldots, n), \\ \displaystyle\int_0^{x_i'} \sigma(x)\, dx > \left(i - \frac{1}{2}\right)\pi - \frac{1}{3 - 2\alpha}\frac{\pi}{2} & (i = 1, 2, \ldots, n) \end{cases} \tag{5}$$

hold, and if $1 \leqq \alpha \leqq 3/2$ *then*

$$(6)\qquad \begin{cases} \displaystyle\int_0^{x_i} \sigma(x)\,dx > i\pi - \frac{\pi}{2} & (i = 1, 2, \dots, n), \\ \displaystyle\int_0^{x_i'} \sigma(x)\,dx > \left(i - \frac{1}{2}\right)\pi - \frac{\pi}{2} & (i = 1, 2, \dots, n). \end{cases}$$

PROOF. Let us introduce the continuous function

$$(7)\qquad \varphi(x) = \operatorname{arctg} \frac{\sigma(x) y(x)}{y'(x)}$$

and let be

$$(8)\qquad \varphi(0) = 0.$$

It is easy to see that this function satisfies the differential equation

$$(9)\qquad \varphi'(x) = \sigma(x) + \frac{\sigma'(x)}{2\sigma(x)} \sin 2\varphi(x).$$

We will prove that the inequality

$$(10)\qquad \varphi'(x) > 0, \ 0 < x < x_n$$

holds. It follows simply from the definition of $\varphi(x)$ that the inequality $0 < \varphi(x) \leqq \pi/2$ is true provided $0 < x \leqq x_1'$ hence (10) is true for $0 < x \leqq x_1'$.

Assuming that (10) does not hold always we define a by

$$a = \min \{x;\ x_1' \leqq x \leqq x_n,\ \varphi'(x) \leqq 0\}.$$

In this case we have

$$(11)\qquad \varphi'(x) > 0$$

for every $0 < x < a$. Since

$$(12)\qquad \begin{cases} \varphi'(a) = \sigma(a) + \dfrac{\sigma'(a)}{2\sigma(a)} \sin 2\varphi(a) = 0, \\ \varphi''(x) = \sigma'(x) + \left[\dfrac{\sigma'(x)}{\sigma(x)}\right]' \dfrac{1}{2} \sin 2\varphi(x) + \dfrac{\sigma'(x)}{\sigma(x)} \cos 2\varphi(x) \cdot \varphi'(x), \end{cases}$$

we obtain, taking into account (3), that

$$\varphi''(a) = \frac{2\sigma'^2(a) - \sigma(a)\sigma''(a)}{\sigma'(a)} > \frac{\beta\sigma'^2(a) - \sigma(a)\sigma''(a)}{\sigma'(a)} \geqq 0.$$

From the continuity of $\varphi''(x)$ it follows that if $a - \varepsilon < x < a$ and ε is sufficiently small, then $\varphi''(x) > 0$ and

$$0 < \int_x^a \varphi''(x)\,dx = -\varphi'(x).$$

Consequently $\varphi'(x) < 0$, which contradicts to (11), therefore (10) is true.

On the basis of (7), (8) and (10) we have

$$\begin{cases} \varphi(x_i) = i\pi & i = 0, 1, 2, \dots, n \\ \varphi(x_i') = \left(i - \dfrac{1}{2}\right)\pi & i = 1, 2, \dots, n. \end{cases} \tag{13}$$

Integrating the differential equation (9) between 0 and x we obtain

$$\varphi(x) = J(x) + F(x) \tag{14}$$

where

$$J(x) = \int_0^x \sigma(u)\, du \quad \text{and} \quad F(x) = \int_0^x \frac{\sigma'(u)}{2\sigma(u)} \sin 2\varphi(u)\, du.$$

It can be easily seen that $F(x)$ takes on its local maxima at $x_1', x_2', \dots, x_n'$ and its local minima at $x_0 = 0, x_1, \dots, x_n$. We will prove that

$$F(x_i') > F(x_{i+1}') \qquad i = 1, 2, \dots, u-1. \tag{15}$$

Now we have taking (9) into account

$$F(x_{i+1}') - F(x_i') = \int_{x_i'}^{x_{i+1}'} \frac{\sigma'(x)}{2\sigma(x)} \sin 2\varphi(x)\, dx =$$

$$= \int_{x_i'}^{x_{i+1}'} \frac{\sin 2\varphi(x)}{\dfrac{2\sigma(x)}{\sigma'(x)} \cdot \varphi'(x)} \varphi'(x)\, dx = \int_{x_i'}^{x_{i+1}'} \frac{\sin 2\varphi(x)}{\dfrac{2\sigma^2(x)}{\sigma'(x)} + \sin 2\varphi(x)} \varphi'(x)\, dx.$$

The function $s(\varphi(x)) \overset{\text{def}}{=} 2\sigma^2(x)/\sigma'(x)$ is steadily increasing because from our restriction for α and from (3)

$$\left[\frac{\sigma^2(x)}{\sigma'(x)}\right]' = \frac{\sigma(x)}{\sigma'^2(x)} [2\sigma'^2(x) - \sigma(x)\sigma''(x)] > 0.$$

So we have

$$F(x_{i+1}') - F(x_i') = \int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} \frac{\sin 2\varphi}{s(\varphi) + \sin 2\varphi}\, d\varphi =$$

$$= \int_{i\pi}^{(i+\frac{1}{2})\pi} \left[\frac{\sin 2\varphi}{s(\varphi) + \sin 2\varphi} - \frac{\sin 2\varphi}{s(\varphi - \frac{\pi}{2}) - \sin 2\varphi} \right] d\varphi =$$

$$= -\int_{i\pi}^{(i+\frac{1}{2})\pi} \sin 2\varphi \frac{\left[s(\varphi) - s\left(\varphi - \frac{\pi}{2}\right)\right] + 2\sin 2\varphi}{[s(\varphi) + \sin 2\varphi]\left[s\left(\varphi - \frac{\pi}{2}\right) - \sin 2\varphi\right]}\, d\varphi < 0$$

hence the inequalities (15) and

$$F(x) \leqq F(x_1'), \quad x \geqq 0 \tag{16}$$

hold. [1]) We obtain another inequality for $F(x_1')$ from (14) by putting $x = x_1'$:

$$\frac{\pi}{2} = J(x_1') + F(x_1') > F(x_1')$$

and $J(x) = \varphi(x) - F(x) \geqq \varphi(x) - F(x_1') > \varphi(x) - \frac{\pi}{2}$. The inequalities (6) follow from this last inequality by putting in it $x = x_i$ and $x = x_i'$ $(i = 1, 2, \ldots, n)$ respectively. To prove the statements (5) a more precise estimation is needed.

Now we want to prove the validity of the inequality

$$F(x_1') < \frac{1}{3-2\alpha} \frac{\pi}{2} \tag{17}$$

if $\alpha < 1$.

This is sufficient for our purpose because from (14) and (16) we have

$$J(x_i) = \varphi(x_i) - F(x_i) > \varphi(x_i) - F(x_1'), \qquad i = 1, 2, \ldots, n$$

and

$$J(x_i') = \varphi(x_i') - F(x_i') \geqq \varphi(x_i') - F(x_1'), \qquad i = 1, 2, \ldots, n,$$

and taking into account (13) and (17) we obtain (5). At first we need the inequality

$$\varphi(x) < x\sigma(x) \quad \text{for} \quad x > 0. \tag{18}$$

From (7) we have $\sigma(x) = \operatorname{tg} \varphi(x) \cdot y'(x)/y(x)$ therefore

$$\lim_{x \to 0} \frac{\varphi(x)}{\sigma(x)} = \lim_{x \to 0} \frac{\varphi(x)}{\operatorname{tg} \varphi(x)} \cdot \frac{y(x)}{y'(x)} = 1 \cdot \frac{0}{y'(0)} = 0. \tag{19}$$

Again, it follows from equation (9), by using the well known inequality $\sin x < x$ for $x > 0$ that

$$\varphi'(x) < \sigma(x) + \frac{\sigma'(x)\varphi(x)}{\sigma(x)}$$

or $[\varphi(x)/\sigma(x)]' < 1$ $(x > 0)$. Integrating this between 0 and x and taking into account (19) we obtain $\varphi(x)/\sigma(x) < x$, i.e. the inequality (18).

We now derive an inequality, which we shall need in proving the inequality (17). If $\sigma(x) > 0$, $\beta < 1$, $\beta\sigma'^2(x) - \sigma(x)\sigma''(x) \geqq 0$ for $a < x < b$ then

$$\int_a^b \sigma(x)\,dx \geqq \sigma(b)(b-a)\frac{1-\beta}{2-\beta}. \tag{20}$$

[1]) In the case $\alpha < 5/4$ there exist inequalities among $F(x_0)$, $F(x_1, \ldots, F(x_n)$ similar to (15). E. Makai proved (see [1]) that with our notations

$$F(x_{i+1}) - F(x_i) = \varphi(x_{i+1}) - \varphi(x_i) - [Y(x_{i+1}) - Y(x_i)] = \pi - \int_{x_i}^{x_{i+1}} \sigma(x)\,dx > 0,$$

hence

$$0 = F(x_0) < F(x_1) < \ldots < F(x_n).$$

Indeed, let us consider the function $t(x) \stackrel{\text{def}}{=} [\sigma(x)]^{1-\beta}$. This function is concave because its second derivative $t''(x) = (1-\beta)\sigma^{-1-\beta}[\sigma\sigma'' - \beta\sigma'^2]$ is not more than 0 by our assumptions, so that

$$t(x) \geqq \frac{t(b)}{b-a}(x-a) \quad \text{for} \quad a \leqq x \leqq b,$$

hence

$$\sigma(x) \geqq \frac{\sigma(b)}{(b-a)^{\frac{1}{1-\beta}}}(x-a)^{\frac{1}{1-\beta}} \quad \text{for} \quad a \leqq x \leqq b.$$

Integrating this over $[a, b]$, we obtain

$$\int_a^b \sigma(x)\,dx \geqq \int_a^b \frac{\sigma(b)}{(b-a)^{\frac{1}{1-\beta}}}(x-a)^{\frac{1}{1-\beta}}\,dx = \frac{1-\beta}{2-\beta}\sigma(b)(b-a)$$

i.e. inequality (20). Since in our case $\beta\sigma'^2(x) - \sigma(x)\sigma''(x) \geqq 0$, where $\beta = 2\alpha - 1 < 1$' $\sigma(x) > 0$ for $0 < x < x_n$, we have from (20) by applying (18) at $x = x_1$

$$J(x_1') = \int_0^{x_1'} \sigma(x)\,dx \geqq \frac{1-\beta}{2-\beta}x_1'\sigma(x_1') > \frac{1-\beta}{2-\beta}\frac{\pi}{2}.$$

It follows from (14) and from the last inequality that

$$F(x_1') = \frac{\pi}{2} - J(x_1') < \frac{1}{2-\beta}\frac{\pi}{2}, \tag{21}$$

which is the same as (17), since $\beta = 2\alpha - 1$.

We mention the interesting case $q''(x) \leqq 0$ when $\alpha = 0$. In this case $F(x_1') < \pi/6$.

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References

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[2] E. Makai, Über Eigenwertabschätzungen bei gewissen homogenen linearen Differentialgleichungen zweiter Ordnung. Compositio Math. **6** (1939), 368—374.

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