Non existence of interpolatory polynomials

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1. Recently Professor P. TURÁN and his associates have written a series of papers on Lacunary interpolation (see [1], [2], [3], [9]). Later these results are generalised in different directions by O. KIS [5, 6], G. FREUD [4], A. SHARMA, and R. B. SAXENA [7, 8] and the author [10, 11, 12, 13]. At present we are concerned with the following theorem of J. SURÁNYI, and P. TURÁN [9].

Let $n=2k+1$ and

$$1 \geqq x_1 > x_2 > \dots > x_k > x_{k+1} = 0 > x_{k+2} > \dots > x_{2k+1} \geqq -1 \tag{1.1}$$

with

$$x_j = -x_{2k+2-j} \qquad (j=k+2, \dots, 2k+1). \tag{1.2}$$

Theorem 1. (J. Surányi and P. Turán) *If $n=2k+1$ and the points $x_1, \dots, x_n$ satisfy* (1.1) *and* (1.2), *there is in general no polynomial $f(x)$ of degree $\leqq 2n-1$ such that for given y_ν and y_ν^**

$$f(x_\nu)=y_\nu,\ f''(x_\nu)=y_\nu^* \qquad (\nu=1, 2, \dots, n). \tag{1.3}$$

If there exists such a polynomial then there is an infinity of them.

Thus as remarked by the above authors in the case of an odd number of distinct symmetrical points x_ν both the problem of existence and uniqueness have a negative solution. Even they have exhibited that in the case of infinitely many solutions in theorem 1 for $n \geqq 3$ the general form of the solution is

$$f(x)=f_0(x)+Cf_1(x) \tag{1.4}$$

where $f_0(x)$ and $f_1(x)$ are fixed polynomials of degree $\leqq 2n-1$ and C arbitrary complex number.

But the situation changes if the number of distinct points $x_1, x_2, \dots, x_n$ is even. More precisely they they proved the following:

Theorem 2. (J. Surányi and P. Turán) *If $n=2k$, then to prescribed values y_ν and y_ν^* there is a uniquely determined polynomial $f(x)$ of degree $\leqq 2n-1$ such that*

$$f(x_\nu)=y_\nu, f''(x_\nu)=y_\nu^* \qquad (\nu=1, 2, \dots, n),$$

if x_ν's stand for the zeros of ultraspherical polynomials.

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The main object of this paper is to construct a simple example of a case of modified (0, 1, 3) lacunary interpolation on Tchebycheff abscissas of the first kind where the polynomial of interpolation does not exist uniquely for either n even or n odd. By modified (0, 1, 3) interpolation we mean that the value, first derivative, and third derivative of the function are prescribed at the zeros of $T_n(x)$ together with the value of $f(x)$ is also prescribed at ± 1.

More precisely we formulate the following:

Theorem 3. *Given a positive integer n and real numbers* y_{k0} $(k=1, 2, \ldots, n+2)$, y_{k1}, y_{k3} $(k=2, 3, \ldots, n+1)$ *there is, in general, no polynomial* $p_{3n+1}(x)$ of degree $\leqq 3n+1$ *such that*

$$(1.\ 5) \qquad p_{3n+1}(x_k)=y_{k0}, \quad k=1, 2, \ldots, n+2.$$

$$(1.\ 6) \qquad p_{3n+1}^{(i)}(x_k)=y_{ki}, \quad k=2, 3, \ldots, n+1, i=1, 3$$

and if there exists such a polynomial then there is an infinity of them.

Here x_k's are taken to be the zeros of $(1-x^2)T_n(x)$, $T_n(x)=\cos n\,\theta$, $\cos\theta=x$.

PROOF. We shall show that in the case if all y_{k0}, y_{k1}, y_{k3} are zero, then there exists a polynomial $p(x)$ of degree $\leqq 3n+1$ (not identically zero) satisfying (1. 5) and (1. 6). Thus it follows from a well known theorem on the solution of a system of equations that in general there does not exist a unique polynomial $p_{3n+1}(x)$ of degree $\leqq 3n+1$ satisfying (1. 5) and (1. 6), and if they exist they are infinitely many.

From the conditions of the theorem, we have

$$(1.\ 7) \qquad p_{3n+1}(x)=(1-x^2)T_n^2(x)q_{n-1}(x)$$

where $q_{n-1}(x)$ is a polynomial in x of degree $\leqq n-1$. As $p_{3n+1}'''(x_k)=0$, $k=2, 3, \ldots, n+1$ we have $-x_k\, q_{n-1}(x_k)+(1-x_k^2)q_{n-1}'(x_k)=0$, for $k=2, 3, \ldots, n+1$. But this means that

$$(1.\ 8) \qquad (1-x^2)q_{n-1}'(x)-xq_{n-1}(x)=CT_n(x)$$

with a numerical C. Let

$$(1.\ 9) \qquad q_{n-1}(x) = \sum_{i=0}^{n-1} C_i \cos i\theta, \quad \cos\theta = x$$

then

$$(1.\ 10) \qquad q_{n-1}'(x) = \sum_{i=1}^{n-1} iC_i \frac{\sin i\theta}{\sin\theta}.$$

Therefore (1. 8) becomes

$$(1.\ 11) \quad -2C_0 \cos\theta + \sum_{i=1}^{n-1} C_i[(i-1)\cos(i-1)\theta-(i+1)\cos(i+1)\theta] = 2C\cos n\theta.$$

Now comparing the co-efficients of $\cos n\theta, \cos(n-1)\theta, \ldots$ and constant term on both sides of (1. 11) we obtain.

$$C_{n-2} = C_{n-4} = \dots = C_4 = C_2 = C_0 = 0 \quad (n \text{ even}) \tag{1. 12}$$

$$C_{n-1} = C_{n-3} = \dots = C_1 = -\frac{2C}{n} \quad (n \text{ even}) \tag{1. 13}$$

$$C_{n-2} = C_{n-4} = \dots = C_3 = C_1 = 0 \quad (n \text{ odd}) \tag{1. 14}$$

$$C_{n-1} = C_{n-3} = \dots = C_4 = C_2 = 2C_0 \quad (n \text{ odd}) \tag{1. 15}$$

Therefore using above relations we obtain

$$p_{3n+1}(x) = C(1-x^2)T_n^2(x) \sum_{k=1}^{\frac{n}{2}} \cos(2k-1)\theta, \qquad n \text{ even}, \tag{1. 16}$$

$$p_{3n+1}(x) = C(1-x^2)T_n^2(x) \left[-\frac{1}{2} + \sum_{k=1}^{\frac{n-1}{2}} \cos 2k\theta \right], \qquad n \text{ odd}. \tag{1. 17}$$

This shows that when n is even or odd there is in general no polynomial $p_{3n+1}(x)$ of degree $\leqq 3n+1$ which satisfies (1. 5) and (1. 6). In other words, there is a universal relationship between the values $\pi_{3n+1}(x_k)$ $(k=1, 2, \dots, n+2)$, $\pi'_{3n+1}(x_k)$ and $\pi'''_{3n+1}(x_k)$ $(k=2, 3, \dots, n+1$, for an arbitrary polynomial $\pi_{3n+1}(x)$ of degree $\leqq 3n+1$, the co-efficients being independent of $\pi_{3n+1}(x)$.

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