

Hereditary subradicals of the lower Baer radical

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1. A (non-empty) class M of (associative) rings is a *radical class* if it satisfies the two conditions:

- (a) Every homomorphic image of a ring in M is again in M .
- (b) If the ring K has the property that every nonzero homomorphic image of K has a nonzero ideal in M , K is in M .

(see [2] for this and related concepts).

If the class M satisfies the condition that every ideal of a ring in M is also in M , then M is said to be *hereditary*.

A class M which satisfies only (a) can always be embedded in a minimal containing radical class as follows [4]; [5]:

Let $M_1 = M$; if M_α is defined for all ordinals $1 \leq \alpha < \beta$, define M_β to consist of all rings K such that every nonzero homomorphic image of K contains a nonzero ideal in M_α for some ordinal $\alpha < \beta$. The class LM consisting of all rings which belong to some M_α is the desired class, called the (*Kurosh*) *lower radical class*.

When M consists of all zero rings (i.e., rings R for which $R^2 = 0$) then $LM = M_2$ [4], and LM is called the *lower Baer radical class*, which we denote by B . If we let Z_0 be the zero ring on the infinite cyclic group and let P consists of all homomorphic images of Z_0 , then $LP = B$ [4]. Thus the lower Baer radical is generated by a single ring, and we write $L(Z_0) = B$. Our purpose is to show that every hereditary radical subclass of B enjoys this property.

2. For a prime p , let Z_p denote the zero ring on the cyclic group of order p .

Theorem 1. Let $R \subseteq B$ a hereditary radical class. Then

(i) $R = B$ if and only if R contains a zero ring having a nonzero element of infinite additive order; hence $R = L(Z_0)$.

(ii) If no zero ring in R has elements of infinite additive order, then $R = L(K)$ where $K = \bigoplus \sum Z_p$, $Z_p \in R$.

Conversely, for any non-empty set Q of (distinct) primes, if $K = \bigoplus \sum_{p \in Q} Z_p$, then LH is a hereditary radical subclass of B , where H consists of all homomorphic images of K .

PROOF. (i) If $R = B$, then $Z_0 \in R$. Conversely, suppose $A \in R$ is a zero ring having an element a of infinite additive order. Since A is a zero ring, every additive subgroup of A is an ideal of A . In particular the cyclic group $\langle a \rangle$ generated by a is an ideal of A , and $\langle a \rangle \cong Z_0$. Thus $Z_0 \in R$ so that $B = L(Z_0) \subseteq R$.

(ii) Assume no zero ring in R has elements of infinite additive order. Note that R contains a zero ring, since if $K \in R$ then $K \in M_2$, so that K contains an ideal I which is a zero ring; hence $I \in R$, since R is hereditary. Then R contains Z_p for some prime p , as follows. If $0 \neq x \in I$, then $px = 0$ for some minimal integer $p > 0$ and, in fact, we can assume p is a prime. Then $(x) \cong Z_p$, (x) is an ideal of R , so $Z_p \in R$. Let Q consist of all distinct primes p for which $Z_p \in R$ and let $K = \bigoplus_{p \in Q} Z_p$ (we choose one representative from each isomorphism class determined by a Z_p). If $\theta(K)$ is a nonzero image of K , then $\theta(Z_p) \neq 0$ for some p , since we are dealing with the direct sum (restricted). Now Z_p is simple so $\theta(Z_p) \cong Z_p$. Thus $K \in R$ by (b). Letting H consist of all homomorphic images of K , we have $L(K) = LH \subseteq R$. On the other hand if $A \in R$ and $\theta(A)$ is a nonzero image of R , then $\theta(A)$ has a nonzero ideal I , with $I^2 = 0$. But then $I \in R$, and as above, every nonzero image of I has an ideal isomorphic to Z_p for some p . Since each $Z_p \in H$, $I \in H_2$ and hence $A \in H_3 \subseteq L(K)$. Thus $R = L(K)$.

Now let $Q = \{p_i\}$ be any non-empty set of distinct primes, enumerated so that $p_i < p_{i+1}$, and let $K = \bigoplus_{p_i \in Q} Z_{p_i}$. We first show that S is a homomorphic image of K if and only if S is isomorphic to an ideal of K . Thus let I be a nonzero ideal of K and let $0 \neq x \in I$. Then x is an infinite-tuple $x = (x_i)$ where $x_i \in Z_{p_i}$ and $x_i = 0$ for all but a finite number of x_i . Choose j minimal so that $x_k = 0$ for all $k > j$. Then $x = (x_1, \dots, x_j, 0, \dots)$ and so $p_1 \dots p_{j-1} x = (0, \dots, p_1 \dots p_{j-1} x_j, \dots) \neq 0$. Now any nonzero element of Z_p generates Z_p additively for any p , so we get $Z_{p_j} \subseteq I$. If $x_{j-1} \neq 0$, after subtraction of a suitable member of Z_{p_j} we repeat to get $Z_{p_{j-1}} \subseteq I$. Thus $Z_{p_i} \subseteq I$ for any $i \leq j$ for which $x_i \neq 0$. Let $A = \{p_j \in Q: \text{the } j^{\text{th}} \text{ component of every } x \in I \text{ is zero}\}$. Then for $p_i \notin A$, $Z_{p_i} \subseteq I$, so $\bigoplus_{p_i \in A} Z_{p_i} \subseteq I$. However I is obviously contained in this sum so that $K \cong I \oplus J$, where J is the sum taken over those $p_i \in A$. Thus $K/I \cong J$ and $K/J \cong I$, and this suffices to establish our claim. In addition, if S is a homomorphic image of K , then since S is a direct sum of Z_p 's, a similar argument shows that any ideal of S is a direct sum of Z_p 's. Thus H consisting of all homomorphic images of K is hereditary. By ([3], Theorem 1) we can assert that $L(K) \subseteq B$ is a hereditary radical class and this completes the proof.

We remark that in case $Q = \{p\}$ then $L(Z_p) = B \cap G_p$, where G_p is the hereditary radical class consisting of all rings whose additive group is p -primary ([1], Lemma 1). We will make use of this for our next result, as well as the following fact from the general theory, which can be found in [2]. If P is a radical class, in any ring A , the sum of all P -ideals of A (i.e., ideals of A belonging to the class P), gives a unique maximal P -ideal, called the P -radical of A , which we denote by $p(A)$.

Let Q be a non-empty set of primes, K, H as in Theorem 1. For a ring A let $h_a(A)$ denote the LH -radical of A , and $h_p(A)$ the $L(Z_p)$ -radical of A for any $p \in Q$. Since $h_p(A) \in G_p$, it follows that for any $p \in Q$, $h_p(A) \cap (\sum_{p \neq q} h_p(A)) = 0$.

Theorem 2. For any ring A , $h_Q(A) = \bigoplus_{p \in Q} h_p(A)$.

PROOF. The above observation implies that the sum of the $h_p(A)$ is direct. Further for each $p \in Q$, $h_p(A) \in L(Z_p) \subseteq LH$, so $h_p(A) \subseteq h_Q(A)$. Then $I = \bigoplus_{p \in Q} h_p(A) \subseteq h_Q(A)$. For the opposite inclusion we first show that if $R \in LH$, then the additive

group of R is torsion. Thus if $t(R)$ is the torsion subgroup of R , then $t(R)$ is an ideal of R and $R/t(R)$ has no nonzero torsion elements. In [1], it was shown that if a class M is hereditary and homomorphically closed then $LM = M_3$. Thus if $R/t(R) \neq 0$, then, since $R/t(R) \in H_3$, it contains an ideal $0 \neq I_2 \in H_2$, and in turn I_2 contains an ideal $0 \neq I_1 \in H_1 = H$. Hence I_1 , being a homomorphic image of $\oplus \Sigma Z_p$, contains at least one Z_p , and this contradicts $t(R/t(R)) = 0$. It now follows that $h_Q(A)$ is a group direct sum of its p -primary components A_p , each of which is an ideal of $h_Q(A)$. It is easily verified that $A_p = h_p(A) \cap h_Q(A)$, since $h_p(A) \in G_p$ and hence each A_p is an ideal of A contained in $h_p(A)$. Thus $h_Q(A) \subseteq I$.

References

- [1] E. P. ARMENDARIZ—W. G. LEAVITT, The hereditary property in the lower radical construction, *Canad. J. Math.* **20** (1968), 474—476.
- [2] N. DIVINSKY, Rings and Radicals, *Toronto*, 1965.
- [3] A. E. HOFFMANN—W. G. LEAVITT, Properties inherited by the lower radical, (to appear).
- [4] A. KUROSU, Radicals of rings and algebras, *Mat. Sbornik* **33** (75) (1953), 13—26 (Russian).
- [5] A. SULINSKI—R. ANDERSON—N. DIVINSKY, Lower radical properties for associative and alternative rings, *J. London Math. Soc.* **41** (1966), 417—424.

(Received February 27, 1967.)