

On a theorem of G. Birkhoff

By B. IMREH (Szeged)

In this note we are going to give a short proof of the following classical theorem of G. BIRKHOFF [1].

Theorem. *A class $\mathfrak{U}$ of Ω -algebras forms a variety if and only if it is closed under forming*

- (1) *subalgebras,*
- (2) *homomorphic images,*
- (3) *direct products.*

Our terminology is that of [2]. However, we mention that in the following by law we mean an arbitrary fully invariant congruence κ of the countably generated Ω -word algebra F . Moreover we say the law κ to hold in the Ω -algebra A if for any congruence φ corresponding to an arbitrary homomorphism of F into A , we have $\varphi \cong \kappa$.

Since the necessity is obvious, we prove only the sufficiency.

If $\mathfrak{U}$ consists only of the one-element algebra, then $\mathfrak{U}$ is the class of all algebras satisfying the law F^2 . If $\mathfrak{U}$ contains an algebra A having more than one element, then A has a subalgebra generated by at most countably many elements of it which also belongs to $\mathfrak{U}$ by (1). Hence there exists a congruence θ in such that $F/\theta \in \mathfrak{U}$. If κ denotes the intersection of all such congruences θ , we shall show that κ is a law.

In fact, if ε is an arbitrary endomorphism of F , then $F\varepsilon$ is a subalgebra of F .

Now consider the union F' , of all classes of the congruence κ which have non-empty intersection with $F\varepsilon$. Then F' is obviously a subalgebra of F , moreover, making use of the second isomorphism theorem, we have

$$(i) \quad F'/\kappa_{F'} \cong F\varepsilon/\kappa_{F\varepsilon}$$

where the set in the subscript indicates the restriction of κ to this set. Now we define the following relation ϱ in F :

$$a\varrho b \quad \text{if and only if} \quad a\varepsilon\kappa b\varepsilon;$$

it is easily seen that ϱ is a congruence in F and

$$(ii) \quad F\varepsilon/\kappa_{F\varepsilon} \cong F/\varrho.$$

By (1) and (3) we have $F/\kappa \in \mathfrak{U}$. Moreover (1) implies $F'/\kappa_{F'} \in \mathfrak{U}$ and thus using (i) and (ii) we obtain $F/\varrho \in \mathfrak{U}$. In view of the definition of κ we have $\kappa \leq \varrho$, hence for any two elements $a, b \in F$ the relation $a\kappa b$ implies $a\varrho b$ i. e. $a\varepsilon\kappa b\varepsilon$ (recall the definition of ϱ). This fact shows that κ is fully invariant, i. e. it is a law.

Now we shall prove that an arbitrary Ω -algebra A belongs to $\mathfrak{U}$ if and only if the law κ holds in it.

Indeed, first assume $A \in \mathfrak{U}$. Let B be an arbitrary subalgebra of A such that $B \cong F/\theta$ holds with a suitable θ . (1) implies $B \in \mathfrak{U}$, so that we have $\theta \cong \kappa$.

Conversely, assume that the law κ holds in A . First, assuming that A is finitely generated, we have $A \cong F/\theta$. By the assumption $\theta \cong \kappa$, hence it follows from the third isomorphism theorem that A is a homomorphic image of F/κ and so (2) implies $A \in \mathfrak{U}$.

In the general case κ holds obviously for any finitely generated subalgebra of A , thus, as it was proved in the preceding paragraph, any such algebra belongs to $\mathfrak{U}$. The finitely generated subalgebras of A form a local system in A , so, making use of a theorem of P. M. COHN [2] (p. 101.), asserting that any class of algebras closed with respect to forming subdirect products and homomorphic images is local, we are led to $A \in \mathfrak{U}$, which completes the proof.

References

- [1] G. BIRKHOFF, On the structure of abstract algebras, *Proc. Cambridge Philos. Soc.* **31** (1935), 433—454.
- [2] P. M. COHN, *Universal algebra*, New York, 1965.

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