

On stars of coverings and uniform spaces

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It is the aim of this note to elucidate some aspects of the interrelation existing between coverings of a set and symmetrical perfect topogenous orders defined on that set. In particular, we shall give a syntopogenous characterization of uniformities based on their definition as systems of coverings.¹⁾

§ 1. Coverings and symmetrical perfect topogenous orders

Let E be a nonvoid set. Coverings of E will be denoted by lower case Greek letters. A cover the members of which are pairwise disjoint will be called a partition. For α, β covers of E , we write

$$\alpha \cap \beta = \{A \cap B \mid A \in \alpha, B \in \beta\},$$

and $\alpha \subseteq \beta$ will mean that for any $A \in \alpha$ there exists a $B \in \beta$ satisfying $A \subseteq B$. — We still need the following

Definition 1. A symmetrical perfect topogenous order on E is a relation $<$ defined on the set of all subsets of E , satisfying the following axioms:

- (01) $\emptyset < \emptyset, E < E$;
- (02) $A < B \Rightarrow A \subseteq B$;
- (03) $A \subseteq A' < B' \subseteq B \Rightarrow A < B$;
- (P') $A_i < B \ (i \in I) \Rightarrow \bigcup \{A_i \mid i \in I\} < B$;
- (S) $A < B \Rightarrow E - B < E - A$. ■²⁾

Remark. (03) and (P') together imply

- (P) $A_i < B_i \ (i \in I) \Rightarrow \bigcup \{A_i \mid i \in I\} < \bigcup \{B_i \mid i \in I\}$.

¹⁾ Of course, the syntopogenous characterization of uniformities based on their definition as systems of coverings can be inferred from two known results, (7.33) in [1] and „Kočetkov's theorem" in [2]. Nevertheless, an explicit formulation of this result and its (nontrivial) direct proof might deserve some interest.

²⁾ As usual, the sign ■ indicates the end of a proof or of a statement.

Again, (P) and (S) together imply

$$(Q) \quad A_i < B_i \ (i \in I) \Rightarrow \cap \{A_i | i \in I\} < \cap \{B_i | i \in I\}. \blacksquare$$

Between covers and symmetrical perfect topogenous orders a close connection is established by the following

Theorem 1. (1) *If γ is a cover of E , then the relation $<_\gamma$ defined between subsets of E by*

$$A <_\gamma B \Leftrightarrow \text{St}(A, \gamma) \subseteq B$$

is a symmetrical perfect topogenous order³⁾ on E .

(2) *For any symmetrical perfect topogenous order $<$ on a set E there exists a cover γ of E such that $< = <_\gamma$.*

(3) *If $<$ and $<_1$ are symmetrical perfect topogenous orders on E satisfying $<_1 \subseteq <$, then there exist covers γ and δ of E , such that $< = <_\gamma$, $<_1 = <_\delta$ and $\gamma \subseteq \delta$.*

On the other hand, $\gamma \subseteq \delta$ always implies $<_\delta \subseteq <_\gamma$.

PROOF. (1) Let γ be a cover of E . Clearly,

$$xUy \Leftrightarrow (\exists G \in \gamma) x, y \in G$$

is a reflexive and symmetrical relation. Put

$$A <_\gamma B \Leftrightarrow \left[\begin{array}{l} x \in A \\ xUy \end{array} \right] \rightarrow y \in B,$$

or equivalently

$$A <_\gamma B \Leftrightarrow \text{St}(A, \gamma) \subseteq B.$$

By [1], (5.41), the relation $<_\gamma$ is a biperfect topogenous order. In view of the implication

$$\text{St}(A, \gamma) \subseteq B \Rightarrow \text{St}(E - B, \gamma) \subseteq E - A$$

it is even symmetrical:

$$A <_\gamma B \Rightarrow E - B <_\gamma E - A.$$

(Of course, it is also possible to check the conditions of Definition 1. directly.)

(2) For $x \in E$ put

$$U(x) = \cap \{V | x < V\} = \{y | x < E - y\}.$$

Clearly, $\{U(x) | x \in E\}$ is a cover of E , symmetrical in the sense that

$$y \in U(x) \Leftrightarrow x \in U(y).$$

Let

$$\gamma = \{H | x \in H \Rightarrow H \subseteq U(x)\}.$$

³⁾ Of course, $\text{St}(A, \gamma) = \cup \{G \in \gamma | G \cap A \neq \emptyset\}$.

For any $x \in E$, $x \in U(x)$ implies $\{x\} \in \gamma$, so γ is a cover of E . Moreover, $\{x, y\} \in \gamma$ for $x \in E$ and $y \in U(x)$. Indeed,

$$\left. \begin{array}{l} x \in U(x) \\ y \in U(x) \end{array} \right\} \Rightarrow \{x, y\} \subseteq U(x),$$

and

$$\left. \begin{array}{l} y \in U(x) \Rightarrow x \in U(y) \\ y \in U(y) \end{array} \right\} \Rightarrow \{x, y\} \subseteq U(y).$$

We see that γ is a cover of E , and also that

$$\text{St}(x, \gamma) = U(x)$$

for any $x \in E$. Now each of the following conditions is equivalent to the next one:

$$\begin{aligned} A < B, \\ x < B \quad (x \in A), \\ U(x) \subseteq B \quad (x \in A), \\ \bigcup \{U(x) \mid x \in A\} \subseteq B, \\ \bigcup \{\text{St}(x, \gamma) \mid x \in A\} \subseteq B, \\ \text{St}(A, \gamma) \subseteq B, \\ A <_\gamma B. \end{aligned}$$

(3) Put $U_1(x) = \bigcap \{V \mid x <_1 V\}$ and

$$\delta = \{H \mid x \in H \Rightarrow H \subseteq U_1(x)\}.$$

One sees that $<_1 = <_\delta$. Moreover, $<_1 \subseteq <$ implies $U(x) \subseteq U_1(x)$ for $x \in E$ and so $H \in U(x)$ implies $H \in U_1(x)$. Hence we get $\gamma \subseteq \delta \Rightarrow \gamma \subseteq \delta$.

Again, if $\gamma \subseteq \delta$, then $\text{St}(A, \gamma) \subseteq \text{St}(A, \delta)$ for any $A \subseteq E$. Thus we have the implication

$$\gamma \subseteq \delta \Rightarrow <_\delta \subseteq <_\gamma. \blacksquare$$

Let us now supplement the results of the previous theorem by a characterization of those orders ⁴⁾ $<_\gamma$ which are generated by partitions γ . One easily sees that the many-to-one correspondence

$$\gamma \Rightarrow <_\gamma$$

becomes one-to-one, if we restrict γ to run through the partitions of E . More detailed information about partition-generated orders $<_\gamma$ is contained in the following

Theorem 2. (1) *A symmetrical perfect topogenous order $<$ is generated by a partition iff*

$$A < B \Rightarrow [A \subseteq H \subseteq B \text{ for some } H \text{ satisfying } H < H].$$

⁴⁾ Unless the contrary is explicitly stated, the words „order” and „relation” stand for „symmetrical perfect topogenous order”.

(2) If a relation $<$ is generated by a partition γ , then this γ is uniquely determined as the class of minimal self-preceding subsets of E :

$$H \in \gamma \text{ iff } H < H \text{ and } \emptyset \neq G \subset H \Rightarrow G \not< H.$$

PROOF. Part (2) is clear; so is the necessity of the condition in (1). The only thing to prove is the sufficiency of that condition: Suppose it holds for an order $<$, and for $x \in E$ put $U(x) = \bigcap \{V | x < V\}$. Clearly $x < U(x)$, and in view of the condition and of the fact that $U(x)$ is the smallest set H satisfying $x < H$, we also have $U(x) < U(x)$. By what has just been said, the implication

$$x < A \Rightarrow U(x) \subseteq A$$

also holds.

Let us now establish the implication

$$(*) \quad U(x) \cap U(y) \neq \emptyset \Rightarrow U(x) = U(y).$$

First of all, $z \in U(x) \Rightarrow x \in U(z)$ $\left. \begin{array}{l} U(z) < U(z) \end{array} \right\} \Rightarrow x < U(z) \Rightarrow U(x) \subseteq U(z)$. By symmetry we get $U(z) \subseteq U(x)$ and finally $z \in U(x) \Rightarrow U(z) = U(x)$.

Let now be $z \in U(x)$ and $z \in U(y)$. Then

$$U(z) = U(x) = U(y)$$

and $(*)$ results proved.

Thus $\gamma = \{U(x) | x \in E\}$ is a partition of E and in the same way as in the proof of Theorem 1. we get $A < B$ iff $\bigcup \{U(x) | x \in A\} \subseteq B$, and this in turn iff $\text{St}(A, \gamma) \subseteq B$. (In establishing the second „iff” we have to make use of the implication

$$a \in U(x) \Rightarrow U(a) = U(x). \blacksquare$$

§ 2. A characterization of uniformities defined as systems of coverings

We start with the following well-known (see e.g. [2] or [3])

Definition 2. A system Σ of coverings of a set E is a uniformity on E if the following conditions are satisfied:

$$(C1) \quad \left. \begin{array}{l} \alpha \in \Sigma \\ \alpha \subseteq \beta \end{array} \right\} \Rightarrow \beta \in \Sigma;$$

$$(C2) \quad \alpha, \beta \in \Sigma \Rightarrow \alpha \cap \beta \in \Sigma;$$

$$(C3) \quad \alpha \in \Sigma \Rightarrow (\exists \beta \in \Sigma) [\text{St}(x, \beta) | x \in E] \subseteq \alpha. \blacksquare$$

The uniformities of a given set are partially ordered in a natural way:

$$\Sigma_1 \subseteq \Sigma_2 \Leftrightarrow (\alpha \in \Sigma_1 \rightarrow \alpha \in \Sigma_2).$$

Remark. ⁵⁾ Condition (C3) can be replaced by the following one:

$$(\quad) \quad \alpha \in \Sigma \Rightarrow (\exists \beta \in \Sigma) [\text{St}(B, \beta) | B \in \beta] \subseteq \alpha.$$

⁵⁾ See e.g. [2], p. 563, bottom. For the readers convenience, we expose in detail the proof outlined there.

PROOF. (C3a) implies (C3), since from $x \in B$ there follows $\text{St}(x, \beta) \subseteq \text{St}(B, \beta)$.

On the other hand, (C3) implies (C3a). As a matter of fact, if to a given $\alpha \in \Sigma$ a cover $\gamma \in \Sigma$ is chosen by (C3), and to this γ a cover $\beta \in \Sigma$ is chosen again by (C3), then β satisfies the requirement of (C3a) with respect to α : If $\{\text{St}(x, \gamma) | x \in E\} \cong \alpha$, and $\{\text{St}(x, \beta) | x \in E\} \cong \gamma$, then for $B \in \beta$ we have

$$\text{St}(B, \beta) = \bigcup \{\text{St}(x, \beta) | x \in B\},$$

and for any $x \in B$,

$$B \subseteq \text{St}(x, \beta) \subseteq G_x \in \gamma.$$

Thus, for any fixed $x_0 \in B$, we can infer from

$$x_0 \in B \subseteq \bigcap \{G_x | x \in B\}$$

that

$$\text{St}(B, \beta) \subseteq \bigcup \{G_x | x \in B\} \subseteq \text{St}(x_0, \gamma). \blacksquare$$

Definition 3. A nonvoid family $\mathcal{S} = \{< | < \in \mathcal{S}\}$ of symmetrical perfect topogenous orders on a set E is a symmetrical perfect syntopogenous structure on E , if it satisfies the following conditions:

$$(S1) \quad <_1, <_2 \in \mathcal{S} \Rightarrow (\exists < \in \mathcal{S})(<_1 \subseteq < \& <_2 \subseteq <);$$

$$(S2) \quad < \in \mathcal{S} \Rightarrow (\exists <' \in \mathcal{S})(< \subseteq <'^2). \blacksquare$$

Remark. Condition (S2) is capable of the following more explicit formulation: If $< \in \mathcal{S}$, then there exists a $<' \in \mathcal{S}$ such that

$$A < B \Rightarrow (\exists C)(A <' C <' B). \blacksquare$$

Definition 4. A symmetrical perfect syntopogenous structure $\mathcal{S}$ on a set E is said to be descending, if the implication

$$(D) \quad \left. \begin{array}{l} < \in \mathcal{S} \\ <_1 \subseteq < \end{array} \right\} \Rightarrow <_1 \in \mathcal{S}$$

holds for any symmetrical perfect topogenous order $<_1$ over E . $\blacksquare$

The symmetrical perfect syntopogenous structures on a given set E can be partially ordered by the following convention:

$$\mathcal{S}_1 \cong \mathcal{S}_2 \Leftrightarrow [<_1 \in \mathcal{S}_1 \Rightarrow (\exists <_2 \in \mathcal{S}_2)(<_1 \subseteq <_2)].$$

For a given order $<$ over E and $x \in E$, write now $U_{<}(x) = \bigcap \{V | x < V\}$ and put $\gamma_{<} = \{U_{<}(x) | x \in E\}$. Clearly, $\gamma_{<}$ is a cover of E . (As a matter of fact, $\gamma_{<}$ will be a cover of E , as soon as $<$ has property (02).) With these notations, there results the following useful

Lemma. $\text{St}(x, \gamma_{<}) = U_{<^2}(x)$ for $x \in E$, and consequently

$$\{\text{St}(x, \gamma_{<}) | x \in E\} = \gamma_{<^2}$$

for any symmetrical perfect topogenous order $<$ on E .

Remark. $<^2$ is a symmetrical perfect topogenous order on E , whenever $<$ is. (See [1], (2. 16), (3. 53) and (4. 23).)

PROOF.

$$U_<(x) = \cap \{V | x < V\} = \{y | x \nless E - y\},$$

and by the symmetry of $<$, $x \nless E - y \Leftrightarrow y \nless E - x$, i.e.

$$y \in U_<(x) \Leftrightarrow x \in U_<(y).$$

Also, $x < U_<(x)$ by the perfectness of $<$, and

$$x < H \Leftrightarrow U_<(x) \subseteq H.$$

Now, each of the following conditions is equivalent to the next one:

$$x <^2 H;$$

$$x < K < H \text{ for some } K \subseteq E;$$

$$x < U_<(x) \subseteq K < H \text{ for some } K \subseteq E;$$

$$x < U_<(x) < H;$$

$$U_<(x) < H;$$

$$y < H \text{ for any } y \in U_<(x);$$

$$U_<(y) \subseteq H \text{ for any } y \in U_<(x);$$

$$\cup \{U_<(y) | y \in U_<(x)\} \subseteq H;$$

$$\cup \{U_<(y) | x \in U_<(y)\} \subseteq H;$$

$$\text{St}(x, \gamma_<) \subseteq H.$$

Thus we have proved

$$x <^2 H \Leftrightarrow \text{St}(x, \gamma_<) \subseteq H,$$

establishing thereby the lemma. ■

Now we are able to characterize ⁶⁾ uniformities defined as systems of coverings by the following

Theorem 3. (1) Let Σ be a uniformity on E . For $\gamma \in \Sigma$ and $A, B \subseteq E$ put $A <_\gamma B \Leftrightarrow \text{St}(A, \gamma) \subseteq B$, and $\mathcal{S}_\Sigma = \{<_\gamma | \gamma \in \Sigma\}$.

The set $\mathcal{S}_\Sigma$ is a descending, symmetrical and perfect syntopogenous structure on E .

(2) Let $\mathcal{S}$ be a descending, symmetrical and perfect syntopogenous structure on E . For $< \in \mathcal{S}$ and $x \in E$ put $U_<(x) = \cap \{V | x < V\}$, and let

$$\gamma_< = \{U_<(x) | x \in E\}.$$

⁶⁾ This characterization of uniformities differs slightly from that given in [1], Chapter 7.: We need a „descending condition” absent in [1]. This difference is due to the fact that here we are dealing with „whole uniformities” and not with (symmetrical) bases for them as is the case in [1].

Let now $\Sigma_{\mathcal{S}}$ be the set of covers of E with some $\gamma_{\prec}$ inscribed:

$$\Sigma_{\mathcal{S}} = \{\gamma \mid \gamma \cong \gamma_{\prec} \text{ for some } \prec \in \mathcal{S}\}.$$

The set $\Sigma_{\mathcal{S}}$ is a uniformity on E .

(3) The mappings $\Sigma \rightarrow \mathcal{S}_{\Sigma}$ and $\mathcal{S} \rightarrow \Sigma_{\mathcal{S}}$ are one-to-one correspondences, inverse to each other, between the sets of all uniformities and all descending symmetrical and perfect syntopogenous structures on E , which preserve the respective partial orders.

PROOF. (1) By Theorem 1. part (1) each relation $\prec_{\gamma}$ is a symmetrical perfect topogenous order on E . Moreover, the set $\mathcal{S}_{\Sigma}$ is descending. As a matter of fact, if γ is a cover of E and $\prec_1 \subseteq \prec_{\gamma}$, then there exists ⁷⁾ a cover δ such that $\prec_1 = \prec_{\delta}$ and $\gamma \cong \delta$.

Let δ be defined as in the proof of part (3) of Theorem 1. Then $\prec_1 = \prec_{\delta}$, and $G \in \gamma$ implies $G \in \delta$. Indeed, if $x \in G$ then $G \subseteq U_1(x)$, and this because in view of $G \subseteq \text{St}(x, \gamma)$ we have the implications

$$y \in G \Rightarrow x \prec_{\gamma} E - y \Rightarrow x \prec_1 E - y \Leftrightarrow x \prec_{\delta} E - y \Rightarrow y \in U_1(x).$$

Thus we have established $\gamma \subseteq \delta$, and thereby also $\gamma \cong \delta$.

We still have to prove that $\mathcal{S}_{\Sigma}$ satisfies the two conditions laid down in Definition 3.

(S1): Let $\gamma, \delta \in \Sigma$. If $\gamma \cong \delta$, then $\prec_{\delta} \subseteq \prec_{\gamma}$ by Theorem 1. part (3).

Now let $\prec_{\gamma_1}, \prec_{\gamma_2} \in \mathcal{S}_{\Sigma}$. Then $\gamma_1, \gamma_2 \in \Sigma$, and this in turn implies $\gamma_1 \cap \gamma_2 \in \Sigma$.

Put $\gamma = \gamma_1 \cap \gamma_2$. Then

$$\gamma \cong \gamma_1 \Rightarrow \prec_{\gamma_1} \subseteq \prec_{\gamma},$$

and

$$\gamma \cong \gamma_2 \Rightarrow \prec_{\gamma_2} \subseteq \prec_{\gamma},$$

i.e. (S1) holds with $\prec_{\gamma} = \prec_{\gamma_1 \cap \gamma_2} \in \mathcal{S}_{\Sigma}$.

(S2): Let $\prec \in \mathcal{S}_{\Sigma}$, i.e. $\prec = \prec_{\gamma}$ for some $\gamma \in \Sigma$. — Choose $\delta \in \Sigma$ so as to have

$$\{\text{St}(K, \delta) \mid K \in \delta\} \cong \gamma.$$

Now let us show that

$$A \prec_{\gamma} B \Rightarrow A \prec_{\delta} \text{St}(A, \delta) \prec_{\delta} B.$$

We clearly have $A \prec_{\delta} \text{St}(A, \delta)$. At the same time, we also have $\text{St}(A, \delta) \prec_{\delta} B$, i.e. we have

$$\left. \begin{array}{l} x \in \text{St}(A, \delta) \\ x, y \in K \in \delta \end{array} \right\} \Rightarrow y \in B.$$

Indeed, $x \in \text{St}(A, \delta)$ implies the existence of a $K_1 \in \delta$ such that $x \in K_1$ and $K_1 \cap A \neq \emptyset$.

Also, we have

$$x, y \in K \subseteq \text{St}(K, \delta) \subseteq G \in \gamma$$

and in view of $x \in K \cap K_1, K_1 \subseteq \text{St}(K, \delta)$.

⁷⁾ This is a somewhat stronger result than the one contained in part (3) of Theorem 1. As a matter of fact, here we have to find a δ corresponding to a fixed γ , whereas in the earlier result only the relation $\prec$ was fixed but not the cover γ satisfying $\prec = \prec_{\gamma}$. (The correspondence $\gamma \mapsto \prec_{\gamma}$ is, of course, many-to-one!)

Now, if $x_1 \in K_1 \cap A$, then $x_1 \in A$ and $x_1, y \in G \in \gamma$.

From $A <_\gamma B$ we now get $y \in B$, and $\text{St}(A, \delta) <_\delta B$ results proved.

(2) We have to check the three conditions of Definition 2.

(C1): Clear.

(C2): We see that for $<_1, <_2 \in \mathcal{S}$ the implication

$$<_1 \subseteq <_2 \Rightarrow [U_{<_2}(x) \subseteq U_{<_1}(x) \ (x \in E)]$$

holds, and consequently $<_1 \subseteq <_2 \Rightarrow \gamma_{<_2} \subseteq \gamma_{<_1}$.

Let now be $\gamma, \delta \in \Sigma_{\mathcal{S}}$. Then $\gamma_{<_1} \subseteq \gamma$ and $\gamma_{<_2} \subseteq \delta$ for some $<_1, <_2 \in \mathcal{S}$. By (S1) there is a $< \in \mathcal{S}$ such that $<_1 \subseteq <$ and $<_2 \subseteq <$. Thus however

$$\left. \begin{array}{l} \gamma_{<} \subseteq \gamma_{<_1} \\ \gamma_{<} \subseteq \gamma_{<_2} \end{array} \right\} \Rightarrow \gamma_{<} \subseteq \gamma_{<_1} \cap \gamma_{<_2} \subseteq \gamma \cap \delta.$$

This shows that $\gamma \cap \delta \in \Sigma_{\mathcal{S}}$.

(C3): Let $\gamma \in \Sigma_{\mathcal{S}}$, i.e. let $\gamma_{<} \subseteq \gamma$ for some $< \in \mathcal{S}$.

Choose now $<_1 \in \mathcal{S}$ in accordance with (S2), i.e. let $< \subseteq <_1^2$. (Of course, $<_1^2 \subseteq <_1 \in \mathcal{S}$ implies $<_1^2 \in \mathcal{S}$, but we do not use this fact.) We see that

$$< \subseteq <_1^2 \Rightarrow \gamma_{<_1^2} \subseteq \gamma_{<},$$

and so, by the Lemma, we obtain

$$\gamma_{<_1^2} = \{\text{St}(x, \gamma_{<_1}) | x \in E\} \subseteq \gamma_{<}.$$

(3) Let $\Sigma \rightarrow \mathcal{S}_{\Sigma}$ and $\mathcal{S} \rightarrow \Sigma_{\mathcal{S}}$. If $\mathcal{S} = \mathcal{S}_{\Sigma}$, then $\Sigma_{\mathcal{S}} = \Sigma$. Indeed,

$$\Sigma_{\mathcal{S}} = \{\gamma | \gamma \supseteq \gamma_{<} \text{ for some } < \in \mathcal{S}\},$$

i.e. in our case

$$\Sigma_{\mathcal{S}} = \{\gamma | \gamma \supseteq \gamma_{<_q} \text{ for some } <_q \in \mathcal{S}_{\Sigma}\} = \{\gamma | \gamma \supseteq \gamma_{<_q} \text{ for some } q \in \Sigma\}.$$

Now $\gamma_{<_q} = \{U_{<_q}(x) | x \in E\}$, and by virtue of

$$U_{<_q}(x) = \cap \{V | x <_q V\} = \cap \{V | \text{St}(x, q) \subseteq V\} = \text{St}(x, q)$$

we get

$$\gamma_{<_q} = \{\text{St}(x, q) | x \in E\}.$$

We see that $\gamma \in \Sigma_{\mathcal{S}}$ for $\mathcal{S} = \mathcal{S}_{\Sigma}$ iff $\{\text{St}(x, q) | x \in E\} \subseteq \gamma$ for some $q \in \Sigma$. This, however, means that $\Sigma_{\mathcal{S}} = \Sigma$. As a matter of fact, for any uniformity Σ we have

$$\alpha \in \Sigma \Leftrightarrow (\exists \beta \in \Sigma) [\{\text{St}(x, \beta) | x \in E\} \subseteq \alpha],$$

the implication $\Rightarrow$ being simply condition (C3), and the reverse implication $\Leftarrow$ being true because by (C1) $\beta \subseteq \{\text{St}(x, \beta) | x \in E\} \subseteq \alpha$ implies $\alpha \in \Sigma$ for any $\beta \in \Sigma$.

Again, let $\mathcal{S} \rightarrow \Sigma_{\mathcal{S}}$ and $\Sigma \rightarrow \mathcal{S}_{\Sigma}$.

If $\Sigma = \Sigma_{\mathcal{S}}$ then $\mathcal{S}_{\Sigma} = \mathcal{S}$.

First of all,

$$\mathcal{S}_{\Sigma} = \{\gamma_{<} | \gamma \in \Sigma\} = \{\gamma_{<} | \gamma \in \Sigma_{\mathcal{S}}\} = \{\gamma_{<} | \gamma \supseteq \gamma_{<} \text{ for some } < \in \mathcal{S}\}.$$

Consider now the subset

$$\{\gamma_{<} | \gamma = \gamma_{<} \text{ for some } < \in \mathcal{S}\}$$

of $\mathcal{S}_\Sigma$, and let $<_1$ be an arbitrary element of this subset, i.e. let there be a $< \in \mathcal{S}$ such that $A <_1 B$ iff $\text{St}(A, \gamma_{<}) \subseteq B$.

Now, each of the following statements is equivalent to the next one:

$$\begin{aligned} \text{St}(A, \gamma_{<}) &\subseteq B, \\ \text{St}(a, \gamma_{<}) &\subseteq B \text{ for } a \in A, \\ U_{<^2}(a) &\subseteq B \text{ for } a \in A, \\ a <^2 B &(a \in A), \\ A <^2 B. \end{aligned}$$

This shows that $<_1 = <^2$, and also that

$$\{<_\gamma | \gamma = \gamma_{<}\} = \{<^2 | < \in \mathcal{S}\}.$$

Now let $\gamma \cong \gamma_{<}$. In view of $\alpha \cong \beta \Rightarrow <_\alpha \subseteq <_\beta$ we have

$$\mathcal{S}_\Sigma = \{<_\gamma | \gamma \cong \gamma_{<} \text{ for some } < \in \mathcal{S}\} = \{<' | <' \subseteq <^2 \text{ for some } < \in \mathcal{S}\} = \mathcal{S}.$$

Indeed, if $<' \in \mathcal{S}$ then $<' \subseteq <^2$ for some $< \in \mathcal{S}$ by condition (S2) from Definition 3. — On the other hand, $<' \subseteq <^2 \subseteq < \in \mathcal{S}$ implies $<' \in \mathcal{S}$ by the descending condition.

Let us still show that the mappings just considered are order-preserving: The implications

$$\Sigma \subseteq \Sigma_1 \Rightarrow \mathcal{S}_\Sigma \subseteq \mathcal{S}_{\Sigma_1} \Rightarrow \mathcal{S}_\Sigma \cong \mathcal{S}_{\Sigma_1}$$

are evident.

On the other hand, $\mathcal{S} \cong \mathcal{S}_1 \Rightarrow \Sigma_\mathcal{S} \subseteq \Sigma_{\mathcal{S}_1}$. Indeed, let $\gamma \in \Sigma_\mathcal{S}$, i.e. let $\gamma \cong \gamma_{<}$ for some $< \in \mathcal{S}$.

By $\mathcal{S} \cong \mathcal{S}_1$ we have $< \subseteq <_1$ for some $<_1 \in \mathcal{S}_1$, and $< \subseteq <_1 \Rightarrow \gamma_{<_1} \cong \gamma_{<}$ now yields

$$\gamma \cong \gamma_{<} \cong \gamma_{<_1} \Rightarrow \gamma \cong \gamma_{<_1}$$

for a $<_1 \in \mathcal{S}_1$, i. e. $\gamma \in \Sigma_{\mathcal{S}_1}$. ■

References

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