

On lattice-ordered algebras with infinitary operations

By O. STEINFELD (Budapest)

Dedicated to Professor A. G. Kuroš on his 60-th birthday

§ 1. Introduction

In the papers [3], [4] L. FUCHS began the systematical investigation on partially ordered (universal) algebras with finitary operations. The purpose of this paper is to introduce a class of lattice-ordered algebras with infinitary operations which play an important rôle in the theory of algebras with infinitary operations.

Let C denote a closure system on a set A . An operation g on A is called C -admissible if every $X \in C$ is closed under it. In Proposition 2.2 it is proved that every C -admissible v -ary operation on A induces a v -ary operation $\bar{g}$ on C such that $\bar{g}$ is *isotone* in its variables. Furthermore $\bar{g}$ is *contractive* in C , i.e. for every $X \in C$

$$\bar{g}(X, X, \dots, X, \dots) \subseteq X$$

holds. (The definitions of the fundamental notions are given in § 2.)

Let $\mathfrak{A} = \langle A; F \rangle$ be an *algebra* (with infinitary or finitary operations). We assume that the meet of all subalgebras of $\mathfrak{A}$ is not void. It is known that the set $S(\mathfrak{A})$ of all subalgebras of $\mathfrak{A}$ forms a closure system on A .

Let $\langle V; G \rangle$ denote an algebra with the following properties:

- (i) V is a lattice under a partial order $\leq$;
- (ii) every fundamental operation $g \in G$ is *isotone* in its variables;
- (iii) every $g \in G$ is *contractive* in V .

By an *algebra-lattice* $\mathfrak{B} = \langle V; G; \leq \rangle$ we mean an algebra $\langle V; G \rangle$ with the properties (i), (ii), (iii). $\mathfrak{B} = \langle V; G; \leq \rangle$ is called a *complete algebra-lattice*, if V is a *complete* lattice under the partial order $\leq$. Making use of the Dedekind—MacNeille completion method we prove that every algebra-lattice is isomorphic to some subalgebra-lattice of a complete algebra-lattice. (See Theorem 5.1.)

In Theorem 2.3 it is proved that every closure system C on a set A forms a complete algebra-lattice under the set inclusion and under the operations $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_n, \dots$ induced by the C -admissible operations $g_0, g_1, \dots, g_n, \dots$ on A . This theorem implies that the set $S(\mathfrak{A})$ of all subalgebras of an algebra $\mathfrak{A}$ forms a complete algebra-lattice under suitable operations. (See Corollary 2.4.)

Conversely, for every complete algebra-lattice $\mathfrak{B} = \langle V; G; \leq \rangle$ one can construct an algebra $\mathfrak{A} = \mathfrak{A}(\mathfrak{B})$ uniquely determined by $\mathfrak{B}$ such that the complete algebra-lattice $S(\mathfrak{A})$ of all subalgebras of $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. (See Theorem 3.2.) The

proof of this representation theorem is based on Lemma 3.1, in which we show that $S(\mathfrak{A})$ coincides with the set of all closed intervals $[0, a]$ ($a \in V$).

A subalgebra $\mathfrak{B}$ of a complete algebra-lattice $\mathfrak{B} = \langle V; G; \equiv \rangle$ is called a *complete subalgebra-ideal* of $\mathfrak{B}$, if $\mathfrak{B}$ is a *complete ideal* of the complete lattice $\langle V; \equiv \rangle$. In Lemma 3.1 it is proved that the set $I(\mathfrak{B})$ of all complete subalgebra-ideals of $\mathfrak{B}$ coincides with the set of all closed intervals $[0, a]$ ($a \in V$) too. Thus one can show that $\mathfrak{B}$ and $I(\mathfrak{B})$ are isomorphic complete algebra-lattices by the mapping

$$a \rightarrow [0, a] \quad (a \in V).$$

(See Theorem 3.3.)

For *algebraic* closure systems on a set and for *finitary algebras* we can sharpen the mentioned results. E.g. the set of all subalgebras of a finitary algebra forms a *compactly generated algebra-lattice* and conversely, for a compactly generated algebra-lattice $\mathfrak{B}$ there exists a finitary algebra, whose subalgebras form a compactly generated algebra-lattice isomorphic to $\mathfrak{B}$. (See Corollary 4.3 and Theorem 4.5. In these results the known lattice-theoretical theorems of BIRKHOFF and FRINK [1] are extended for algebra-lattices.)

In § 6 we define the *complete grupoid-lattice* which is a special complete algebra-lattice. Example 6.2 (6.4) shows that the set of all subgroups of a group (the set of all subrings of a ring) forms a complete grupoid-lattice under suitable operations. Examples 6.8 and 6.9 are applications of Corollary 2.4 and Theorem 3.2.

§ 2. On algebras and complete algebra-lattices

Let A be a non-void set and let v be an ordinal number. A v -ary operation on A is a function $f(x_0, x_1, \dots, x_\eta, \dots)$ ($\eta < v$) which associates with every sequence $a_0, a_1, \dots, a_\eta, \dots$ ($\in A; \eta < v$) an element $f(a_0, a_1, \dots, a_\eta, \dots)$ ($\eta < v$) of A . If f is a (v -ary) operation on the set A , then we say that A is *closed under the operation* f . In the case $v = n < \omega_0$, a n -ary operation is called *finitary*. A 0-ary (*nullary*) operation picks out a certain distinguished element in A . The v -ary operations, where $v \geq \omega_0$ are called *infinitary*.

Let τ be an ordinal number. A sequence $\mathfrak{A} = \langle A; f_0, f_1, \dots, f_\xi, \dots \rangle$ ($\xi < \tau$) in which A is a non-void set and f_ξ is a v_ξ -ary operation on A is called an *algebra of type* $A = \langle v_0, v_1, \dots, v_\xi, \dots \rangle$ ($\xi < \tau$). τ is called the *order of* A . In the algebra $\mathfrak{A} = \langle A; f_0, f_1, \dots, f_\xi, \dots \rangle$ A denotes the *underlying set* and $f_0, f_1, \dots, f_\xi, \dots$ are the *fundamental operations on* A . Two algebras of the same type are called *similar*. (We use mostly the notions and the terminology of J. SLOMINSKY [9]).

Let $\mathfrak{A} = \langle A; f_0, f_1, \dots, f_\xi, \dots \rangle$ be an algebra of type $A = \langle v_0, v_1, \dots, v_\xi, \dots \rangle$ ($\xi < \tau$) of order τ and B a non-void subset of A . If the fundamental operations $f_0, f_1, \dots, f_\xi, \dots$ ($\xi < \tau$) are also operations on B , that is if B is closed under each of the operations $f_0, f_1, \dots, f_\xi, \dots$ ($\xi < \tau$), then the sequence $\mathfrak{B} = \langle B; f_0, f_1, \dots, f_\xi, \dots \rangle$ ($\xi < \tau$) is also an algebra of type A . $\mathfrak{B}$ is called a *subalgebra of algebra* $\mathfrak{A}$. We also write $\langle A; F \rangle$ for $\mathfrak{A} = \langle A; f_0, f_1, \dots, f_\xi, \dots \rangle$ and $\langle B; F \rangle$ for the subalgebra $\langle B; f_0, f_1, \dots, f_\xi, \dots \rangle$ of $\mathfrak{A}$.

Let $S(\mathfrak{A})$ denote the set of all subalgebras $\mathfrak{B}_\gamma = \langle B_\gamma; F \rangle$ ($\gamma \in \Gamma$) of an algebra $\mathfrak{A} = \langle A; F \rangle$. If $\mathfrak{B}_\delta = \langle B_\delta; F \rangle$ ($\delta \in \Delta \subseteq \Gamma$) is a family of subalgebras of $\mathfrak{A}$, then the set-theoretical intersection $\bigcap_{\delta \in \Delta} \mathfrak{B}_\delta = \langle \bigcap_{\delta \in \Delta} B_\delta; F \rangle$ of the subalgebras $\mathfrak{B}_\delta$ is either the

void set or a subalgebra of $\mathfrak{A}$. In this paper we assume that the intersection of all subalgebras of an algebra $\mathfrak{A}$ is always a subalgebra of $\mathfrak{A}$. Thus the set $S(\mathfrak{A})$ of all subalgebras of an algebra $\mathfrak{A}$ forms a complete lattice under the set-theoretical inclusion.

If X is a non-void subset of the underlying set A of an algebra $\mathfrak{A} = \langle A; F \rangle$ and $\mathfrak{B}_\delta (\delta \in \Delta)$ is the family of all subalgebras of $\mathfrak{A}$ which contain X , then the intersection $\bigcap_{\delta \in \Delta} \mathfrak{B}_\delta$ is the subalgebra of $\mathfrak{A}$ generated by X . This subalgebra will be denoted by $\{X\}$.

Let A be an arbitrary set and $B(A)$ the set of all its subsets. A subset C of $B(A)$ is said to be a *closure system* if C is closed under intersections, i.e. for any subsystem $D \subseteq C$, we have $\bigcap D \in C$.

Note that $A \in C$, by the definition of the intersection of a void family of subsets. Since a closure system C admits arbitrary intersections, it is easy to prove that C forms a complete lattice (with respect to the set-theoretical inclusion). However, it need not be a sublattice of $B(A)$. (See P. M. COHN [2] Ch. II. 1.)

Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra. The complete lattice $S(\mathfrak{A})$ of all subalgebras of $\mathfrak{A}$ is e.g. a closure system of A .

A *closure operator* on a partially ordered set $\langle P; \equiv \rangle$ is a mapping Φ of P into itself with the properties:

$$(2.1) \quad \text{if } x \equiv y, \text{ then } \Phi(x) \equiv \Phi(y),$$

$$(2.2) \quad x \equiv \Phi(x),$$

$$(2.3) \quad \Phi(\Phi(x)) = \Phi(x)$$

for all $x, y \in P$.

For a closure operator Φ the element $\Phi(x)$ is called the Φ -closure of x ; if an element x coincides with its Φ -closure, it is said to be Φ -closed.

The following theorem is proved in COHN [2].

Theorem 2.1. (See [2] Ch. II. Theorem 1.1). *Every closure system C on a set A defines a closure operator Φ on the complete lattice $B(A)$ of all subsets of A by the rule*

$$\Phi(X) = \bigcap_{X \subseteq Y \in C} Y \quad (X \in B(A)).$$

Conversely, every closure operator Φ on $B(A)$ defines a closure system C on A by

$$C = \text{the set of all } X \in B(A) \text{ with } \Phi(X) = X,$$

and the correspondence $C \leftrightarrow \Phi$ between closure systems and closure operators thus defined is bijective.

Let us consider a closure system C of a set A . We call an operation g on A *C-admissible* if every set $X \in C$ is closed under the operation g .

Let g be an arbitrary v -ary operation on the set A which is C -admissible with respect to the closure system C on A . For a sequence $A_0, A_1, \dots, A_\eta, \dots \in C (\eta < v)$ the set of all elements $g(a_0, a_1, \dots, a_\eta, \dots)$ where a_η is an arbitrary element of $A_\eta (\eta < v)$ is denoted by $g(A_0, A_1, \dots, A_\eta, \dots)$. Let Φ denote the closure operator

on $B(A)$ corresponding to C , determined by Theorem 2.1. We define the operation $\bar{g}$ on the set C by the equality

$$(2.4) \quad \bar{g}(A_0, A_1, \dots, A_\eta, \dots) = \Phi(g(A_0, A_1, \dots, A_\eta, \dots)) \quad (\eta < \nu).$$

We call $\bar{g}$ the operation on C induced by g . (Cf. L. FUCHS [4]).

Let $S = S(\mathfrak{A})$ denote the closure system of all subalgebras of an algebra $\mathfrak{A} = \langle A; F \rangle$. In view of Theorem 2.1 the closure operator Φ determined by $S = S(\mathfrak{A})$ has the form

$$\Phi(X) = \bigcap_{X \subseteq Y \in S(\mathfrak{A})} Y = \{X\} \quad (X \in S(\mathfrak{A})).$$

Let g be an arbitrary S -admissible ν -ary operation on A . For a sequence $\mathfrak{B}_0, \mathfrak{B}_1, \dots, \dots, \mathfrak{B}_\eta, \dots (\eta < \nu)$ of subalgebras of $\mathfrak{A}$ the operation $\bar{g}$ on $S(\mathfrak{A})$ induced by g is defined by

$$(2.4') \quad \bar{g}(\mathfrak{B}_0, \mathfrak{B}_1, \dots, \mathfrak{B}_\eta, \dots) = \{g(\mathfrak{B}_0, \mathfrak{B}_1, \dots, \mathfrak{B}_\eta, \dots)\} \quad (\eta < \nu).$$

Let $\langle P; \equiv \rangle$ be a partially ordered set. A ν -ary operation f on P is *isotone in its variables* $x_0, x_1, \dots, x_\eta, \dots (\eta < \nu)$ if the inequalities $a_\eta \equiv b_\eta$ ($a_\eta, b_\eta \in P; 0 \leq \eta < \nu$) imply the inequality

$$f(a_0, a_1, \dots, a_\eta, \dots) \equiv f(b_0, b_1, \dots, b_\eta, \dots) \quad (\eta < \nu)$$

for each sequence $a_0, a_1, \dots, a_\eta, \dots \in P$.

A μ -ary operation g on P is called *contractive* in P if

$$(2.5) \quad g(\overset{0}{a}, \overset{1}{a}, \dots, \overset{\xi}{a}, \dots) \equiv a \quad (\xi < \mu)$$

holds for every element a of P .

Proposition 2.2. Let C be a closure system on a set A and g a C -admissible ν -ary operation on A . Then the ν -ary operation $\bar{g}$ on C induced by g has the following properties:

- (a) $\bar{g}$ is isotone in its variables,
- (b) $\bar{g}$ is contractive in C .

PROOF. Because of (2.1) and (2.4) the operation g has evidently the property

- (a). As g is a C -admissible operation on A , for every $B \in C$

$$\bar{g}(\overset{0}{B}, \overset{1}{B}, \dots, \overset{\eta}{B}, \dots) = \Phi(g(\overset{0}{B}, \overset{1}{B}, \dots, \overset{\eta}{B}, \dots)) \subseteq \Phi(B) = B.$$

holds, where Φ denotes the closure operator on $B(A)$ corresponding to C . Thus $\bar{g}$ is contractive in C .

We can now define a class of partially ordered algebraic systems which plays an important rôle in this paper.

Let $\langle V; f_0, f_1, \dots, f_\xi, \dots \rangle$ be an algebra of type $A = \langle v_0, v_1, \dots, v_\xi, \dots \rangle$ ($\xi < \tau$) of order τ with the following properties:

- (i) in V a partial order $\equiv$ is defined such that V forms a lattice under it. (The lattice operations are denoted by $\wedge$ and $\vee$);

(ii) every fundamental operation $f_0, f_1, \dots, f_\xi, \dots (\xi < \tau)$ is isotone in its variables;

(iii) every $f_0, f_1, \dots, f_\xi, \dots (\xi < \tau)$ is contractive in V .

By an *algebra-lattice* $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ we mean an algebra $\langle V; f_0, f_1, \dots, f_\xi, \dots \rangle$ of type $A = \langle v_0, v_1, \dots, v_\xi, \dots \rangle (\xi < \tau)$ of order τ , satisfying the properties (i), (ii), (iii). (An algebra-lattice is a special lattice-ordered algebra with infinitary operations; for the definition of lattice-ordered algebras with finitary operations see in FUCHS [3] p. 117.)

If we assume instead of (i) the condition

(i') in V a partial order $\equiv$ is defined such that V forms a complete lattice under it. (The lattice operations are denoted by $\wedge$ and $\vee$), then the algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ is called *complete*.

Naturally a complete algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ has a *least element* $0 = \bigwedge_{x \in V} x$ and a *greatest element* $e = \bigvee_{x \in V} x$.

With the help of Proposition 2.2 we can prove

Theorem 2.3. *Let C be a closure system on a set A , σ an ordinal number and $G = \langle g_0, g_1, \dots, g_\eta, \dots \rangle (\eta < \sigma)$ a sequence of C -admissible operations on A . If $\bar{g}_\eta$ is the operation on C induced by $g_\eta (\eta < \sigma)$, then C forms a complete algebra-lattice under the operations $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_\eta, \dots (\eta < \sigma)$ and under the set inclusion $\subseteq$.*

PROOF. If $g_0, g_1, \dots, g_\eta, \dots (\eta < \sigma)$ are μ_0 -ary, μ_1 -ary, $\dots, \mu_\eta$ -ary, $\dots (\eta < \sigma)$ operations on A , then in view of Proposition 2.2 $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_\eta, \dots (\eta < \sigma)$ are μ_0 -ary, μ_1 -ary, $\dots, \mu_\eta$ -ary, $\dots$ operations on C . Thus $\mathfrak{C} = \langle C; \bar{g}_0, \bar{g}_1, \dots, \bar{g}_\eta, \dots \rangle (\eta < \sigma)$ is an algebra of type $\langle \mu_0, \mu_1, \dots, \mu_\eta, \dots \rangle$ of order σ . We have to show that the algebra $\mathfrak{C}$ satisfies the condition (i'), (ii), (iii). Evidently $\mathfrak{C}$ is a complete lattice under the set-theoretical inclusion. As every operation $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_\eta, \dots (\eta < \sigma)$ has the properties (a) and (b), therefore conditions (ii) and (iii) are fulfilled.

Since the set $S(\mathfrak{A})$ of all subalgebras of an algebra $\mathfrak{A} = \langle A; F \rangle$ is a closure system on A , Theorem 2.3 implies the following

Corollary 2.4. *Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra and $S = S(\mathfrak{A})$ the closure system of all subalgebras of $\mathfrak{A}$. If σ is an ordinal number and $G = \langle g_0, g_1, \dots, g_\eta, \dots \rangle (\eta < \sigma)$ a sequence of S -admissible operations on A , then S forms a complete algebra-lattice under the operations $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_\eta, \dots$ and under the set inclusion, where $\bar{g}_\eta$ is the operation on S induced by $g_\eta (\eta < \sigma)$.*

Remark 1. It is possible that two S -admissible operations f, g of an algebra $\mathfrak{A}$ are different but the operations $\bar{f}, \bar{g}$ on the set $S(\mathfrak{A})$ induced by f and g are identical. E.g. let R be an associative ring and let

$$f(a, b) = a + b, \quad g(a, b) = a + b - ab \quad (a, b \in R)$$

be two admissible binary operations on R . If A and B are two arbitrary subrings of R , then in view of (2.4')

$$\bar{f}(A, B) = \{a + b; a \in A, b \in B\} \quad \text{and} \quad \bar{g}(A, B) = \{a + b - ab; a \in A, b \in B\}$$

are two operations on the set $S(R)$ of all subrings of R . The elements $f(a, b)$

and $g(a, b)$ of R are generally not equal, but $\bar{f}(A, B)$ and $\bar{g}(A, B)$ both are the subring of R generated by A and B .

Remark 2. Evidently not all the identities of the algebra $\mathfrak{A} = \langle A; f_0, f_1, \dots, \dots, f_\xi, \dots \rangle$ ($\xi < \tau$) retain their validity in the algebra $\langle S(\mathfrak{A}); \bar{f}_0, \bar{f}_1, \dots, \bar{f}_\xi, \dots \rangle$ where $\bar{f}_\xi$ ($\xi < \tau$) is the operation on $S(\mathfrak{A})$ induced by f_ξ . (See Example 6. 3).

§ 3. A representation theorem on complete algebra-lattices

Let $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ ($\xi < \tau$) be a complete algebra-lattice. $\mathfrak{M} = \langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ ($\xi < \tau$) is a *complete subalgebra-lattice* of $\mathfrak{B}$, if $\mathfrak{M}$ is a subalgebra and a complete sublattice of $\mathfrak{B}$.

We call a (non-void) subset A of a complete lattice $\mathfrak{Q} = \langle L; \wedge, \vee \rangle$ a *complete ideal* of $\mathfrak{Q}$, if the following two conditions hold:

- (1) if a_λ ($a_\lambda \in A; \lambda \in \Lambda$) denotes an arbitrary system of elements of A , then $\bigvee_{\lambda \in \Lambda} a_\lambda \in A$;
 (2) if $a \in A$ and $x \in L$, then $a \wedge x \in A$.

Evidently every complete ideal of a complete lattice $\mathfrak{Q}$ is a complete sublattice of $\mathfrak{Q}$.

A complete subalgebra-lattice $\mathfrak{M} = \langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ of a complete algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\eta, \dots; \wedge, \vee \rangle$ is called a *complete subalgebra-ideal* of $\mathfrak{B}$, if $\langle W; \wedge, \vee \rangle$ is a complete ideal of the complete lattice $\langle V; \wedge, \vee \rangle$.

In Corollary 2. 4 we give a method how one can obtain complete algebra-lattices from a given algebra. Now we shall define to every complete algebra-lattice $\mathfrak{B}$ a uniquely determined algebra which plays an important rôle in the representation of $\mathfrak{B}$.

Let $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ be an arbitrary complete algebra-lattice. We define for every element a of V the following unary operation g_a on V :

$$(3.1) \quad g_a(x) = \begin{cases} a, & \text{if } a \vee x = x, \\ 0, & \text{if } a \vee x \neq x \end{cases} \quad (x \in V).$$

Let $g_a, g_b, \dots$ ($a, b, \dots \in V$) be a sequence of all unary operations defined by (3.1). We call the algebra $\mathfrak{A} = \mathfrak{A}(\mathfrak{B}) = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots \rangle$ the *algebra belonging to the complete algebra-lattice $\mathfrak{B}$* . (If α is the cardinality of V and $\alpha \equiv \aleph_0$, then we consider the meet $\wedge$ and join $\vee$ as infinitary operations on V .)

Lemma 3. 1. Let $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ be a complete algebra-lattice. For a non-void subset W of V the following conditions are equivalent:

- (a) $\mathfrak{M} = \langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ is a complete subalgebra-ideal of $\mathfrak{B}$;
 (b) $0 \in W$ and $\mathfrak{M} = \langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ is a convex¹⁾ complete subalgebra-lattice of $\mathfrak{B}$;
 (c) W has the form $W = [0, w]$, where $w = \bigvee_{x \in W} x$;
 (d) $\mathfrak{B} = \langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots \rangle$ is a subalgebra of the algebra $\mathfrak{A} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots \rangle$ belonging to $\mathfrak{B}$.

¹⁾ A subset of a partially ordered set is called *convex* if it contains the whole interval $[a, b]$ whenever it contains the endpoints a, b .

PROOF. The implication (a) $\Rightarrow$ (b) follows immediately from the property (2).

(b) $\Rightarrow$ (c). As W is closed under the operation $\vee$, the element $w = \bigvee_{x \in W} x$ belongs

to W . Thus $0, w \in W$ and the convexity of W imply (c).

(c) $\Rightarrow$ (d). Let us assume that $W = [0, w]$ ($w = \bigvee_{x \in W} x$) holds. We have to prove

that W is closed under every fundamental operation $f_0, f_1, \dots, f_\xi, \dots, \wedge, \vee, g_a, g_b, \dots$ ($a, b, \dots \in V$) on V . Evidently $W = [0, w]$ is closed under the operations $\wedge$ and $\vee$. By the conditions (ii) and (iii) the operations $f_0, f_1, \dots, f_\xi, \dots$ are isotone in their variables and they are contractive in V , therefore $[0, w]$ ($w \in V$) is closed under these operations too. Let us consider an arbitrary unary operation g_k ($k \in V$). If for the element y of the closed interval $[0, w]$ the relation $k \vee y = y$ holds, then $0 \leq k \leq y \leq w$ and so because of (3. 1)

$$g_k(y) = k \in [0, w].$$

If z is an element of $[0, w]$ with the property $k \vee z \neq z$, then in view of (3. 1)

$$g_k(z) = 0 \in [0, w]$$

holds. Thus the interval $W = [0, w]$ is closed under each operation $g_k \in V$.

(d) $\Rightarrow$ (a). Let $\mathfrak{B} = \langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots \rangle$ be a subalgebra of the algebra $\mathfrak{A} = \mathfrak{A}(\mathfrak{B}) = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots \rangle$ belonging to $\mathfrak{B}$. As the underlying set W is closed under the operations $f_0, f_1, \dots, f_\xi, \dots, \wedge$ and $\vee$, therefore $\mathfrak{B} = \langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ is a complete subalgebra-lattice of $\mathfrak{B}$. We have only to show that $\langle W; \wedge, \vee \rangle$ is a complete ideal of the complete lattice $\langle V; \wedge, \vee \rangle$. By the definition of $\mathfrak{B}$ the property (1) holds evidently. If $a \in W$ and $x \in V$, then for the element $b = a \wedge x$

$$g_b(a) = b = a \wedge x \quad (b \vee a = a)$$

holds. As W is closed under the unary operation g_b , therefore $a \wedge x = b = g_b(a) \in W$. Thus the property (2) holds too.

With the help of Lemma 3.1 we can prove our main result. First we must define some fundamental notions.

Let $\mathfrak{A} = \langle A; f_0, f_1, \dots, f_\xi, \dots \rangle$ and $\mathfrak{B} = \langle B; f'_0, f'_1, \dots, f'_\xi, \dots \rangle$ be two similar algebras of type $A = \langle v_0, v_1, \dots, v_\xi, \dots \rangle$ ($\xi < \tau$) of order τ . Let us assume that φ is a one-to-one mapping of A onto B such that

$$\varphi(f_\xi(a_0, a_1, \dots, a_\eta, \dots)) = f'_\xi(\varphi(a_0), \varphi(a_1), \dots, \varphi(a_\eta), \dots) \quad (\xi < \tau; \eta < v_\xi)$$

holds for each pair of fundamental operations f_ξ, f'_ξ and for all sequences $a_0, a_1, \dots, \dots, a_\eta, \dots (\in A; \eta < v_\xi)$, then we say that the algebras $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic by the isomorphism* φ .

We call the (complete) algebra-lattices $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ and $\mathfrak{B}' = \langle W; f'_0, f'_1, \dots, f'_\xi, \dots; \wedge', \vee' \rangle$ *isomorphic*, if the similar algebras $\langle V; f_0, f_1, \dots, f_\xi, \dots \rangle$ and $\langle W; f'_0, f'_1, \dots, f'_\xi, \dots \rangle$ furthermore the (complete) lattices $\langle V; \wedge, \vee \rangle$ and $\langle W; \wedge', \vee' \rangle$ are isomorphic by the same isomorphism φ .

Theorem 3. 2. Let $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ ($\xi < \tau$) be a complete algebra-lattice and let $\mathfrak{A} = \mathfrak{A}(\mathfrak{B}) = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots \rangle = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots \rangle$ be the algebra belonging to $\mathfrak{B}$. Let

$\langle S = S(\mathfrak{A}); \cap, \cup \rangle$ denote the complete lattice of all subalgebras of $\mathfrak{A}$. Then $\mathfrak{B}$ is isomorphic to the complete algebra-lattice $\mathfrak{S} = \langle S; \bar{f}_0, \bar{f}_1, \dots, \bar{f}_\xi, \dots; \cap, \cup \rangle$, where $\bar{f}_\xi$ is the operation on $S = S(\mathfrak{A})$ induced by f_ξ ($\xi < \tau$).

PROOF. From the equivalence of the conditions (c) and (d) in Lemma 3.1 we obtain that the set $S = S(\mathfrak{A})$ of all subalgebras of the algebra $\mathfrak{A} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots \rangle$ is exactly the set of all closed intervals $[0, a]$ ($a \in V$). If $f_0, f_1, \dots, \dots, f_\xi, \dots$ ($\xi < \tau$) are v_0 -ary, v_1 -ary, $\dots$, v_ξ -ary, $\dots$ operations on V , then the induced operations $\bar{f}_0, \bar{f}_1, \dots, \bar{f}_\xi, \dots$ ($\xi < \tau$) are v_0 -ary, v_1 -ary, $\dots$, v_ξ -ary, $\dots$ operations on $S(\mathfrak{A})$. Namely, for an arbitrary sequence $[0, a_0], [0, a_1], \dots, [0, a_\eta], \dots$ ($a_0, a_1, \dots, \dots, a_\eta, \dots \in V; \eta < v_\xi$) of subalgebras of $\mathfrak{A}$

$$(3.2) \quad \bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots) = \{f_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots)\} \\ (\eta < v_\xi)$$

holds, according to (2.4'). In view of Corollary 2.4 $\mathfrak{S} = \langle S; \bar{f}_0, \bar{f}_1, \dots, \bar{f}_\xi, \dots; \cap, \cup \rangle$ is a complete algebra-lattice.

Now we will show that the given complete algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \dots, \dots, f_\xi, \dots; \wedge, \vee \rangle$ ($\xi < \tau$) is isomorphic to $\mathfrak{S}$. First we remark that the algebras $\langle V; f_0, f_1, \dots, f_\xi, \dots \rangle$ and $\langle S; \bar{f}_0, \bar{f}_1, \dots, \bar{f}_\xi, \dots \rangle$ both are of the same type $\Delta = \langle v_0, v_1, \dots, v_\xi, \dots \rangle$ ($\xi < \tau$) of order τ . It is enough to prove that the mapping

$$(3.3) \quad \varphi: a \rightarrow [0, a] \quad (a \in V; [0, a] \in S(\mathfrak{A}))$$

of V onto $S(\mathfrak{A})$ is an isomorphism between the algebras $\langle V; f_0, f_1, \dots, f_\xi, \dots \rangle$, $\langle S(\mathfrak{A}); \bar{f}_0, \bar{f}_1, \dots, \bar{f}_\xi, \dots \rangle$ and between the complete lattices $\langle V; \wedge, \vee \rangle, \langle S(\mathfrak{A}); \cap, \cup \rangle$. Naturally φ is a one-to-one mapping of V onto $S(\mathfrak{A})$. Thus we have to show only the following properties of homomorphism:

$$(3.4) \quad \text{for every sequence } a_0, a_1, \dots, a_\eta, \dots (\in V; \eta < v_\xi)$$

$$f_\xi(a_0, a_1, \dots, a_\eta, \dots) \rightarrow [0, f_\xi(a_0, a_1, \dots, a_\eta, \dots)] = \bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots);$$

$$(3.5) \quad \text{for every system of elements } a_\lambda (\in V; \lambda \in A) \text{ the relations}$$

$$\bigwedge_{\lambda \in A} a_\lambda \rightarrow [0, \bigwedge_{\lambda \in A} a_\lambda] = \bigcap_{\lambda \in A} [0, a_\lambda] \quad \text{and} \quad \bigvee_{\lambda \in A} a_\lambda \rightarrow [0, \bigvee_{\lambda \in A} a_\lambda] = \bigcup_{\lambda \in A} [0, a_\lambda]$$

hold.

Because of the definition of the operation $\bar{f}_\xi$

$$(3.6) \quad f_\xi(a_0, a_1, \dots, a_\eta, \dots) \in \bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots) \quad (\eta < v_\xi).$$

As $\bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots)$ is a subalgebra of $\mathfrak{A}$, therefore it must have the form

$$\bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots) = [0, b] \quad (b \in V).$$

Thus (3.6) implies $[0, f_\xi(a_0, a_1, \dots, a_\eta, \dots)] \subseteq \bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots)$. Conversely, according to (3.2) the subalgebra $\bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots)$ is generated by all the elements $f_\xi(x_0, x_1, \dots, x_\eta, \dots)$ ($x_0 \in [0, a_0], x_1 \in [0, a_1], \dots, \dots, x_\eta \in [0, a_\eta], \dots$ ($\eta < v_\xi$)). Since f_ξ is isotone in its variables, we have

$$0 \subseteq f_\xi(x_0, x_1, \dots, x_\eta, \dots) \subseteq f_\xi(a_0, a_1, \dots, a_\eta, \dots) \quad (0 \subseteq x_\eta \subseteq a_\eta; \eta < v_\xi),$$

whence $\bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots) \subseteq [0, f_\xi(a_0, a_1, \dots, a_\eta, \dots)]$ ($\eta < v_\xi$) follows. Thus (3. 4) is proved.

Let us show that the relation (3. 5) is true. Namely, if x is an element of the subalgebra $[0, \bigwedge_{\lambda \in A} a_\lambda]$ then $0 \leq x \leq \bigwedge_{\lambda \in A} a_\lambda \leq a_\lambda$ ($\lambda \in A$) holds, whence $x \in \bigcap_{\lambda \in A} [0, a_\lambda]$. On the other hand, if the element y is contained in every subalgebra $[0, a_\lambda]$ ($\lambda \in A$), then $y \leq \bigwedge_{\lambda \in A} a_\lambda$. Hence $y \in [0, \bigwedge_{\lambda \in A} a_\lambda]$.

Because $\bigcup_{\lambda \in A} [0, a_\lambda]$ denotes the subalgebra of $\mathfrak{A}$ generated by the subalgebras $[0, a_\lambda]$ ($\lambda \in A$), therefore $\bigvee_{\lambda \in A} a_\lambda \in \bigcup_{\lambda \in A} [0, a_\lambda]$. On the other hand the subalgebra $\bigcup_{\lambda \in A} [0, a_\lambda]$ of $\mathfrak{A}$ has the form $\bigcup_{\lambda \in A} [0, a_\lambda] = [0, u]$ with a suitable element $u \in V$, whence $[0, \bigvee_{\lambda \in A} a_\lambda] \subseteq \bigcup_{\lambda \in A} [0, a_\lambda]$. Conversely, every subalgebra $[0, a_\lambda]$ ($\lambda \in A$) of $\mathfrak{A}$ is contained in the subalgebra $[0, \bigvee_{\lambda \in A} a_\lambda]$, therefore $\bigcup_{\lambda \in A} [0, a_\lambda] \subseteq [0, \bigvee_{\lambda \in A} a_\lambda]$ must hold.

Thus (3. 5₂) is proved and the proof of Theorem 3. 2 is finished.

Let us consider a complete algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$. It follows from Lemma 3. 1 that the set $I(\mathfrak{B})$ of all complete subalgebra-ideals of $\mathfrak{B}$ coincides with the set of all closed intervals $[0, a]$ ($a \in V$). Hence $I(\mathfrak{B})$ is a closure system on V such that the operations $f_0, f_1, \dots, f_\xi, \dots$ are $I(\mathfrak{B})$ -admissible. If f_ξ is a v_ξ -ary operation and $[0, a_0], [0, a_1], \dots, [0, a_\eta], \dots$ ($\eta < v_\xi$; $a_0, a_1, \dots, a_\eta, \dots \in V$) is an arbitrary sequence of complete subalgebra-ideals of $\mathfrak{B}$, then because of (2. 4) and the properties of f_ξ the operation $\bar{f}_\xi$ on $I(\mathfrak{B})$ induced by f_ξ satisfies the relation

$$(3. 7) \quad \bar{f}_\xi([0, a_0], [0, a_1], \dots, [0, a_\eta], \dots) = [0, f_\xi(a_0, a_1, \dots, a_\eta, \dots)] \quad (\eta < v_\xi).$$

In view of Theorem 2. 3 $\mathfrak{T} = \langle I(\mathfrak{B}); f_0, f_1, \dots, f_\xi, \dots; \cap, \cup \rangle$ is a complete algebra-lattice. Lemma 3. 1 and Theorem 3. 2 imply that $\mathfrak{B}$ is isomorphic to $\mathfrak{T}$ by the mapping

$$(3. 8) \quad a \rightarrow [0, a] \quad (a \in V).$$

So we have obtained:

Theorem 3. 3. Let $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ ($\xi < \tau$) be a complete algebra-lattice and $I(\mathfrak{B})$ the set of all complete subalgebra-ideals of $\mathfrak{B}$. By the mapping (3. 8) $\mathfrak{B}$ is isomorphic to the complete algebra-lattice $\mathfrak{T} = \langle I(\mathfrak{B}); \bar{f}_0, \bar{f}_1, \dots, \bar{f}_\xi, \dots; \cap, \cup \rangle$, where $\bar{f}_\xi$ is the operation on $I(\mathfrak{B})$ induced by f_ξ ($\xi < \tau$).

We mention the following special case of Theorem 3. 3.

Corollary 3. 4. Let L be a complete lattice. The set of all complete ideals of L forms under the set theoretical inclusion a complete lattice such that is isomorphic to L by the mapping (3. 8).

§ 4. On finitary algebras and compactly generated algebra-lattices

An algebra $\mathfrak{A} = \langle A; f_0, f_1, \dots, f_\xi, \dots \rangle$ of type $\Delta = \langle n_0, n_1, \dots, n_\xi, \dots \rangle$ ($\xi < \tau$) of order τ is called *finitary*, if $n_0, n_1, n_\xi, \dots$ are non-negative integers. For finitary algebras we can sharpen Theorems 2. 3 and 3. 2 and Corollary 2. 4.

An element a of a complete lattice $\langle L; \wedge, \vee \rangle$ is called *compact* if the following condition is satisfied: if $a \leq \bigvee_{\gamma \in \Gamma} x_\gamma$ ($x_\gamma \in L$), then there exists a finite subset $\Gamma' (\subseteq \Gamma)$ such that $a \leq \bigvee_{\gamma \in \Gamma'} x_\gamma$.

By a *compactly generated algebra-lattice* $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ we mean a finitary algebra $\langle V; f_0, f_1, \dots, f_\xi, \dots \rangle$ of type $\Delta = \langle n_0, n_1, \dots, n_\xi, \dots \rangle$ ($\xi < \tau$) of order τ satisfying the properties (i'), (ii), (iii) and

(iv) every element b of the complete lattice $\langle V; \wedge, \vee \rangle$ can be written as $b = \bigvee_{\mu} a_\mu$, where all the a_μ are compact elements of V , i.e. $\langle V; \wedge, \vee \rangle$ is a *compactly generated lattice*.

A closure system C on a set A is called *algebraic (or inductive)* if every chain in C has a supremum in C .

The following result is known. (See G. GRÄTZER [6] Chapter 0, § 6, Theorem 4.)

Theorem 4.1. *Every algebraic closure system C on a set A forms a compactly generated lattice with respect to the set inclusion.*

Theorems 2.3 and 4.1 imply

Theorem 4.2. *Let C be an algebraic closure system on a set A , σ an ordinal number and $G = \langle g_0, g_1, \dots, g_\eta, \dots \rangle$ ($\eta < \sigma$) a sequence of C -admissible finitary operations on A . If $\bar{g}_\eta$ is the operation on C induced by g_η ($\eta < \sigma$), then C forms a compactly generated algebra-lattice under the operations $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_\eta, \dots$ ($\eta < \sigma$) and under the set inclusion.*

In P. M. COHN [2] it is proved that the set $S(\mathfrak{A})$ of all subalgebras of a finitary algebra $\mathfrak{A}$ is an algebraic closure system. (See [2] Theorem 5.2.) BIRKHOFF—FRINK [1] proved directly that $S(\mathfrak{A})$ forms a compactly generated lattice under the set inclusion.²⁾

These results, Theorem 4.2 and Corollary 2.4 imply

Corollary 4.3. *Let $\mathfrak{A} = \langle A; F \rangle$ be a finitary algebra and $S = S(\mathfrak{A})$ the algebraic closure system of all subalgebras of $\mathfrak{A}$. If σ is an ordinal number and $G = \langle g_0, g_1, \dots, g_\eta, \dots \rangle$ ($\eta < \sigma$) is a sequence of S -admissible finitary operations on A , then $\langle S; \bar{g}_0, \bar{g}_1, \dots, \bar{g}_\eta, \dots; \cap, \cup \rangle$ is a compactly generated algebra-lattice, where $\bar{g}_\eta$ is the operation on S induced by g_η ($\eta < \sigma$).*

Let K denote the set of all compact elements of a compactly generated lattice $\mathfrak{Q} = \langle L; \wedge, \vee \rangle$. Evidently the zero element 0 of $\mathfrak{Q}$ is always compact. As the union of two compact elements is a compact element, $\langle K; 0, \vee \rangle$ forms a semi-lattice with 0 under the nullary operation 0 and the binary operation $\vee$. We define for every element k of K the following unary operation g_k on K :

$$(4.1) \quad g_k(x) = \begin{cases} k, & \text{if } k \vee x = x, \\ 0, & \text{if } k \vee x \neq x \end{cases} \quad (x \in K).$$

Let $g_k, g_l, \dots$ ($k, l, \dots \in K$) be a sequence of all unary operations defined by (4.1).

²⁾ This theorem is generalized in G. GRÄTZER [5] for the subalgebras of an algebra (with infinitary operations).

We call the finitary algebra $\mathfrak{A} = \mathfrak{A}(K) = \langle K; 0, \vee, g_k, g_l, \dots \rangle$ of type $\langle 0, 2, 1, 1, \dots \rangle$ the *finitary algebra belonging to K*.

An *ideal* of a semilattice $\mathfrak{S} = \langle F; \vee \rangle$ is a non-void set A of F such that, for all $a, b \in F$

$$a \vee b \in A \quad \text{if and only if} \quad a, b \in A.$$

We prove the following analogon of Lemma 3.1.

Lemma 4.4. (Cf. GRÄTZER—SCHMIDT [7]). *Let K denote the set of all compact elements of the compactly generated lattice $\mathfrak{Q} = \langle L; \wedge, \vee \rangle$. For a non-void subset B of K the following conditions are equivalent:*

(a') B is an ideal of the semilattice $\langle K; \vee \rangle$;

(c') B has the form $B = [0, b] \cap K = [0, b]_K$, where $b = \bigvee_{y \in B} y$;

(d') $\mathfrak{B} = \langle B; 0, \vee, g_k, g_l, \dots \rangle$ is a subalgebra of the finitary algebra $\mathfrak{A} = \mathfrak{A}(K) = \langle K; 0, \vee, g_k, g_l, \dots \rangle$ belonging to K .

PROOF. (a') $\Rightarrow$ (c'). If B is an ideal of $\langle K; \vee \rangle$, then for the element $b = \bigvee_{y \in B} y$

$$B \subseteq [0, b] \cap K = [0, b]_K$$

holds. To show that $[0, b]_K = B$, we must prove that $x \in [0, b]_K$ implies $x \in B$. If $x \in [0, b]_K$, then x is a compact element with the property

$$x \leq b = \bigvee_{y \in B} y.$$

Hence, $x \leq \bigvee_{y \in B'} y$ for some finite subset B' of B . Set $B' = \langle y_1, y_2, \dots, y_n \rangle$ and $b' = y_1 \vee y_2 \vee \dots \vee y_n$. Then $x \vee b' = b' \in B$ and the ideal property of B imply $x \in B$, which was to be proved.

(c') $\Rightarrow$ (d'). Let $B = [0, b] \cap K = [0, b]_K$ with $b = \bigvee_{y \in B} y$. Evidently $0 \in [0, b]_K$ and $[0, b]_K$ is closed under the binary operation $\vee$. Now let us consider an arbitrary unary operation g_k ($k \in K$). (4.1) implies for every element y of $[0, b]_K = B$ that $g_k(y) \in [0, b]_K$ holds. Thus $\langle B; 0, \vee, g_k, g_l, \dots \rangle$ is a subalgebra of $\langle K; 0, \vee, g_k, g_l, \dots \rangle$, indeed.

(d') $\Rightarrow$ (a'). Now let $\mathfrak{B} = \langle B; 0, \vee, g_k, g_l, \dots \rangle$ be a subalgebra of $\mathfrak{A} = \mathfrak{A}(K) = \langle K; 0, \vee, g_k, g_l, \dots \rangle$. If $x, y \in B$, then evidently $x \vee y \in B$. Conversely, let us assume that for the elements $k, l \in K$

$$k \vee l \in B$$

holds. We have to show that $k, l \in B$. Since B is closed under the operations g_k and g_l ($k, l \in K$), it follows from (4.1)

$$g_k(k \vee l) = k \in B \quad \text{and} \quad g_l(k \vee l) = l \in B.$$

Thus B is an ideal of the semilattice $\langle K; \vee \rangle$.

For the compactly generated algebra-lattices, Theorem 3.2 has the following analogon:

Theorem 4.5. *Let $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ be a compactly generated algebra-lattice and let K denote the set of all compact elements of the complete lattice $\langle V; \wedge, \vee \rangle$. The set $S(\mathfrak{A})$ of all subalgebras of the finitary algebra $\mathfrak{A} = \mathfrak{A}(K)$ belonging*

to K forms under suitable operations a compactly generated algebra-lattice isomorphic to $\mathfrak{B}$.

PROOF. First we shall show that the mapping

$$(4.2) \quad a \rightarrow [0, a]_K = [0, a] \cap K \quad (a \in V)$$

is one-to-one. Let us assume that for the elements $a, b \in V$ the relation $[0, a]_K = [0, b]_K$ holds. Because of the condition (iv) one can write the element $a \in V$ in the form $a = \bigvee_{\lambda \in A} a_\lambda$, where $a_\lambda \in K$. This implies $a_\lambda \in [0, a] \cap K = [0, a]_K$ for every $\lambda \in A$, whence we obtain

$$a = \bigvee_{\lambda \in A} a_\lambda \leq \bigvee_{x \in [0, a]_K} x \leq a,$$

that is $a = \bigvee_{x \in [0, a]_K} x$. Similarly $b = \bigvee_{y \in [0, b]_K} y$. Thus the assumption $[0, a]_K = [0, b]_K$ implies $a = b$.

In view of Lemma 4.4 we obtained that the set $S(\mathfrak{A})$ of all subalgebras of the finitary algebra $\mathfrak{A} = \langle K; 0, \vee, g_k, g_l, \dots \rangle$ belonging to K is exactly the set of all intersections $[0, a]_K = [0, a] \cap K$ ($a \in V$). It is known that $S(\mathfrak{A})$ forms a compactly generated lattice under the set-theoretical inclusion. Now we have to define the „suitable” operations on $S(\mathfrak{A})$. If the operations ³⁾ $f_0, f_1, \dots, f_\xi, \dots$ ($\xi < \tau$) on V are n_0 -ary, n_1 -ary, $\dots$, n_ξ -ary, $\dots$, then we can define the following n_0 -ary, n_1 -ary, $\dots$, n_ξ -ary, $\dots$ operations $f'_0, f'_1, \dots, f'_\xi, \dots$ ($\xi < \tau$) on $S(\mathfrak{A})$. For an arbitrary sequence $[0, a_1]_K, [0, a_2]_K, \dots, [0, a_{n_\xi}]_K$ ($a_1, a_2, \dots, a_{n_\xi} \in V$) of subalgebras of $\mathfrak{A}$ let

$$(4.3) \quad f'_\xi([0, a_1]_K, [0, a_2]_K, \dots, [0, a_{n_\xi}]_K) = [0, f_\xi(a_1, a_2, \dots, a_{n_\xi})]_K.$$

It is obvious that the operation f'_ξ on $S(\mathfrak{A})$ defined by (4.3) is isotone in its variables and it is contractive in $S(\mathfrak{A})$.

Denoting by $\cap$ and $\cup$ the lattice-operations in $S(\mathfrak{A})$ we obtained that $\mathfrak{S} = \langle S(\mathfrak{A}); f'_0, f'_1, \dots, f'_\xi, \dots; \cap, \cup \rangle$ is a compactly generated algebra-lattice, furthermore $\langle V; f_0, f_1, \dots, f_\xi, \dots \rangle$ and $\langle S(\mathfrak{A}); f'_0, f'_1, \dots, f'_\xi, \dots \rangle$ ($\xi < \tau$) are finitary algebras of the same type $\langle n_0, n_1, \dots, n_\xi, \dots \rangle$ of order τ . We shall show that the mapping (4.2) is an isomorphism of the given compactly generated algebra-lattice $\mathfrak{B}$ onto $\mathfrak{S} = \langle S(\mathfrak{A}); f'_0, f'_1, \dots, f'_\xi, \dots; \cap, \cup \rangle$. As we have seen (4.2) is a one-to-one mapping.

For the operation f'_ξ the property of homomorphism $f'_\xi(a_1, a_2, \dots, a_{n_\xi}) \rightarrow [0, f'_\xi(a_1, a_2, \dots, a_{n_\xi})]_K = f'_\xi([0, a_1]_K, [0, a_2]_K, \dots, [0, a_{n_\xi}]_K)$ ($a_1, a_2, \dots, a_{n_\xi} \in V$) follows from (4.3).

To complete the proof, we have to show the relations

$$(4.4) \quad \bigwedge_{\lambda \in A} a_\lambda \rightarrow [0, \bigwedge_{\lambda \in A} a_\lambda]_K = \bigcap_{\lambda \in A} [0, a_\lambda]_K \quad \text{and} \quad \bigvee_{\lambda \in A} a_\lambda \rightarrow [0, \bigvee_{\lambda \in A} a_\lambda]_K = \bigcup_{\lambda \in A} [0, a_\lambda]_K$$

for an arbitrary system a_λ ($\lambda \in A$) of elements of V . The proof of (4.4) is analogous to the proof of (3.5).

It follows from Lemma 4.4 and the proof of Theorem 4.5 the following analogon of Theorem 3.3.

³⁾ Since the operations $f_0, f_1, \dots, f_\xi, \dots$ ($\xi < \tau$) generally are not $S(\mathfrak{A})$ -admissible, we cannot use Corollary 4.3.

Theorem 4.6. Let $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ ($\xi < \tau$) be a compactly generated algebra-lattice and let $\langle K; \vee \rangle$ denote the semilattice of all compact elements of $\langle V; \wedge, \vee \rangle$. If $I(K)$ denotes the set of all ideals of $\langle K; \vee \rangle$, then by the mapping (4.2) $\mathfrak{B}$ is isomorphic to the compactly generated algebra-lattice $\langle I(K); f'_0, f'_1, \dots, f'_\xi, \dots; \cap, \cup \rangle$, where the operation f'_ξ ($\xi < \tau$) is defined by (4.3).

This theorem is an extension of the mentioned result of GRÄTZER and SCHMIDT [7] for the compactly generated algebra-lattices.

§ 5. Embedding of algebra-lattices into complete algebra-lattices

We shall prove the following

Theorem 5.1. Every algebra-lattice is isomorphic to some subalgebra-lattice of a suitable complete algebra-lattice.

This theorem is an analogon of the known result of MACNEILLE [8] about the embedding of lattices into complete lattices. (See e.g. G. SZÁSZ [10] Corollary of Theorem 27).

WARD [11] proved the following result.

Proposition 5.2. Let Φ be a closure operator of a complete lattice L . The set Z_Φ of the Φ -closed elements of L forms a complete lattice under the operations

$$\inf_{Z_\Phi} R = \inf_L R, \quad \sup_{Z_\Phi} R = \Phi(\sup_L R)$$

for an arbitrary subset R of Z_Φ . (See e.g. G. SZÁSZ [10] Theorem 25.).

For a subset X of a partially ordered set $\langle P; \leq \rangle$ an upper (lower) bound of X in P is an element $u \in P$ ($v \in P$) such that $x \leq u$ ($x \geq v$) for every $x \in X$. The set of all upper (lower) bounds of X will be denoted by $U(X)$ ($L(X)$). The set $L(U(X)) = D(X)$ is called the Dedekind cut determined by X . It is easy to show that the following relations hold:

$$(5.1) \quad X \subseteq Y (\subseteq P) \text{ implies } U(X) \supseteq U(Y) \text{ and } L(X) \supseteq L(Y),$$

$$(5.2) \quad X \subseteq L(U(X)) \text{ and } X \subseteq U(L(X)),$$

$$(5.3) \quad U(L(U(X))) = U(X) \text{ and } L(U(L(X))) = L(X).$$

One gets from (5.1), (5.2) and (5.3):

$$(5.4) \quad \text{if } X \subseteq Y (\subseteq P), \text{ then } D(X) = L(U(X)) \subseteq L(U(Y)) = D(Y),$$

$$(5.5) \quad X \subseteq D(X) = L(U(X)) \quad (X \subseteq P),$$

$$(5.6) \quad D(D(X)) = L[U(L(U(X)))] = L(U(X)) = D(X) \quad (X \subseteq P).$$

As the set $B(P)$ of all subsets of the partially ordered set P forms a complete lattice under the set-theoretical inclusion, we have verified the following known result:

Proposition 5.3. Let $B(P)$ denote the complete lattice of all subsets of the partially ordered set P . The mapping

$$X \rightarrow L(U(X)) = D(X) \quad (X \in B(P))$$

of $B(P)$ into itself is a closure operator on $B(P)$.

Now we begin the proof of Theorem 5.1. Let $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee \rangle$ be an arbitrary algebra-lattice. If $B(\mathfrak{B})$ denotes the complete lattice of all subsets of the lattice $\langle V; \wedge, \vee \rangle$, then in view of Proposition 5.3 the mapping

$$X \rightarrow L(U(X)) = D(X) \quad (X \in B(\mathfrak{B}))$$

of $B(\mathfrak{B})$ into itself is a closure operator on $B(\mathfrak{B})$. Using Proposition 5.2 for the closure operator D on the complete lattice $B(\mathfrak{B})$ we get that the set Z_D of all D -closed elements of $B(\mathfrak{B})$ is a complete lattice under the operations of $\inf_{Z_D}$ and $\sup_{Z_D}$, defined in Proposition 5.2. We shall show that Z_D forms under suitable operations a complete algebra-lattice, in which $\mathfrak{B}$ can be embedded isomorphically.

Let us consider the fundamental operation f_ξ of V . If f_ξ is a v_ξ -ary operation, then we consider an arbitrary sequence $A_0, A_1, \dots, A_\eta, \dots \in Z_D$ ($\eta < v_\xi$). Let $f_\xi(A_0, A_1, \dots, A_\eta, \dots)$ denote the set of all elements $f_\xi(a_0, a_1, \dots, a_\eta, \dots)$ ($a_0 \in A_0, a_1 \in A_1, \dots, a_\eta \in A_\eta, \dots$). We define by

$$(5.7) \quad \begin{aligned} f_\xi^*(A_0, A_1, \dots, A_\eta, \dots) &= L(U(f_\xi(A_0, A_1, \dots, A_\eta, \dots))) = \\ &= D(f_\xi(A_0, A_1, \dots, A_\eta, \dots)) \quad (\eta < v_\xi) \end{aligned}$$

a v_ξ -ary operation⁴⁾ on Z_D . We are going to show that $\langle Z_D; f_0^*, f_1^*, \dots, f_\xi^*, \dots; \inf_{Z_D}, \sup_{Z_D} \rangle$ is a complete algebra-lattice.

If $A_0, A_1, \dots, A_\eta, \dots \in Z_D$ and $B_0, B_1, \dots, B_\eta, \dots \in Z_D$ are two sequences with the property $A_\eta \subseteq B_\eta$ ($0 \leq \eta < v_\xi$), then

$$f_\xi(A_0, A_1, \dots, A_\eta, \dots) \subseteq f_\xi(B_0, B_1, \dots, B_\eta, \dots) \quad (\eta < v_\xi).$$

This implies by (5.4) and (5.7)

$$f_\xi^*(A_0, A_1, \dots, A_\eta, \dots) \subseteq f_\xi^*(B_0, B_1, \dots, B_\eta, \dots) \quad (\eta < v_\xi).$$

So f_ξ^* is isotone in its variables, i.e. property (ii) is fulfilled for the operation f_ξ^* .

Now let us consider an arbitrary D -closed set $A (\in Z_D)$ and let $u (\in V)$ be an upper bound of A . If $f_\xi(a_0, a_1, \dots, a_\eta, \dots)$ ($a_0, a_1, \dots, a_\eta, \dots \in A$) denotes an arbitrary element of the set $f_\xi(\overset{0}{A}, \overset{1}{A}, \dots, \overset{\eta}{A}, \dots)$ ($\eta < v_\xi$), then because of the conditions (ii) and (iii) given for f_ξ

$$f_\xi(a_0, a_1, \dots, a_\eta, \dots) \leq f_\xi(\overset{0}{u}, \overset{1}{u}, \dots, \overset{\eta}{u}, \dots) \leq u \quad (\eta < v_\xi)$$

holds. This means that every upper bound $u (\in V)$ of A is an upper bound of the set $f_\xi(\overset{0}{A}, \overset{1}{A}, \dots, \overset{\eta}{A}, \dots)$, that is

$$U(A) \subseteq U(f_\xi(\overset{0}{A}, \overset{1}{A}, \dots, \overset{\eta}{A}, \dots)) \quad (\eta < v_\xi).$$

⁴⁾ In (5.7) we extend the definition (8) of FUCHS [3] for infinitary operations.

This implies because of (5.1), (5.7) and $D(A) = L(U(A)) = A$

$$\begin{aligned} f_{\xi}^*(\overset{0}{A}, \overset{1}{A}, \dots, \overset{\eta}{A}, \dots) &= D(f_{\xi}(\overset{0}{A}, \overset{1}{A}, \dots, \overset{\eta}{A}, \dots)) = \\ &= L(U(f_{\xi}(\overset{0}{A}, \overset{1}{A}, \dots, \overset{\eta}{A}, \dots))) \subseteq L(U(A)) = D(A) = A, \end{aligned}$$

that is the operation f_{ξ}^* is contractive in Z_D .

Thus $\langle Z_D; f_0^*, f_1^*, \dots, f_{\xi}^*, \dots; \inf_{Z_D}, \sup_{Z_D} \rangle$ is a complete algebra-lattice indeed.

Now we show that the mapping

$$(5.8) \quad x \rightarrow D(x) = L(x) \quad (x \in V)$$

is an isomorphism of the algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \dots, f_{\xi}, \dots; \wedge, \vee \rangle$ into the complete algebra-lattice

$$\mathfrak{Z} = \langle Z_D; f_0^*, f_1^*, \dots, f_{\xi}^*, \dots; \inf_{Z_D}, \sup_{Z_D} \rangle.$$

Evidently (5.8) is a one-to-one mapping. One can verify the laws of homomorphism

$$\begin{aligned} x \wedge y \rightarrow D(x \wedge y) &= \inf_{Z_D}(D(x), D(y)) \quad (x, y \in V), \\ x \vee y \rightarrow D(x \vee y) &= \sup_{Z_D}(D(x), D(y)) \quad (x, y \in V) \end{aligned}$$

by the usual method. (See e.g. the proof of Theorem 27 in G. SZÁSZ [10]). We have only to prove the relation

$$\begin{aligned} (5.9) \quad f_{\xi}(x_0, x_1, \dots, x_{\eta}, \dots) \rightarrow D(f_{\xi}(x_0, x_1, \dots, x_{\eta}, \dots)) &= \\ = f_{\xi}^*(D(x_0), D(x_1), \dots, D(x_{\eta}), \dots) \quad (\eta < v_{\xi}) \end{aligned}$$

for an arbitrary sequence $x_0, x_1, \dots, x_{\eta}, \dots \in V$. Because of $L(x) = D(x) \in Z_D$ and (5.7) we have

$$\begin{aligned} (5.10) \quad f_{\xi}^*(D(x_0), D(x_1), \dots, D(x_{\eta}), \dots) &= f_{\xi}^*(L(x_0), L(x_1), \dots, L(x_{\eta}), \dots) = \\ &= D(f_{\xi}(L(x_0), L(x_1), \dots, L(x_{\eta}), \dots)) = L(U(f_{\xi}(L(x_0), L(x_1), \dots, L(x_{\eta}), \dots))) \quad (\eta < v_{\xi}). \end{aligned}$$

As $f_{\xi}(x_0, x_1, \dots, x_{\eta}, \dots)$ is the greatest element of the set $f_{\xi}(L(x_0), L(x_1), \dots, L(x_{\eta}), \dots)$ therefore we have

$$U(f_{\xi}(L(x_0), L(x_1), \dots, L(x_{\eta}), \dots)) = U(f_{\xi}(x_0, x_1, \dots, x_{\eta}, \dots)) \quad (\eta < v_{\xi}).$$

This and (5.10) imply

$$\begin{aligned} f_{\xi}(x_0, x_1, \dots, x_{\eta}, \dots) \rightarrow D(f_{\xi}(x_0, x_1, \dots, x_{\eta}, \dots)) &= L(U(f_{\xi}(x_0, x_1, \dots, x_{\eta}, \dots))) = \\ &= L(U(f_{\xi}(L(x_0), L(x_1), \dots, L(x_{\eta}), \dots))) = f_{\xi}^*(D(x_0), D(x_1), \dots, D(x_{\eta}), \dots) \end{aligned}$$

that is (5.9) is true.

The proof of Theorem 5.1 is finished.

Remark 3. By the embedding of the algebra-lattice $\mathfrak{B}$ into the complete algebra-lattice $\mathfrak{Z}$ the identities between operations of $\mathfrak{B}$ carry over only exceptionally to $\mathfrak{Z}$. In FUCHS [3] a sufficient condition is given for an identity to carry over to $\mathfrak{Z}$.

§ 6. Examples and applications

First we define a special complete algebra-lattice which plays an important rôle in the following examples.

Let L be a *partially ordered groupoid*, i.e. let in L a binary multiplication and a partial order $\cong$ be defined with the property

$$(6.1) \quad a \cong b (a, b \in L) \text{ implies } ac \cong bc \text{ and } ca \cong cb \text{ for all } c \in L.$$

We assume that the following conditions hold in L :

$$(6.2) \quad a^2 \cong a \quad (\text{for all } a \in L)$$

and L is a *complete lattice* with respect to the partial order $\cong$, whose *least*, *greatest elements* are 0 and e , respectively, satisfying

$$(6.3) \quad 0 \cdot e = e \cdot 0 = 0.$$

The partially ordered groupoid L with the mentioned properties is called a *complete groupoid-lattice*.

From the conditions (6.1), (6.3) it follows

$$(6.4) \quad 0 \cdot a = a \cdot 0 = 0 \quad (\text{for all } a \in L).$$

Evidently every complete groupoid-lattice L is a special complete algebra-lattice, as properties (i'), (ii) and (iii) are fulfilled in L .

Example 6.1. Let $\mathfrak{Q}_0 = \langle C_0; \cdot, 0 \rangle$ be a groupoid with 0. If A, B are two subgroupoids with zero of $\mathfrak{Q}_0$, then in view of (2.4') the multiplication defined on C_0 induces the following binary multiplication

$$(6.5) \quad AB = \{ab; a \in A, b \in B\}$$

on the set L_1 of all subgroupoids with zero of $\mathfrak{Q}_0$. It is easy to show that L_1 forms a complete groupoid-lattice under the multiplication (6.5) and under the set inclusion. We denote by $\cap$ and $\cup$ the lattice operations in the complete lattice L_1 . Evidently 0 is the least element and C_0 is the greatest one of L_1 .

Example 6.2. Let L_2 denote the set of all subgroups of a group G . It is known that L_2 forms a complete lattice under the set inclusion. Evidently the commutator-forming $[x, y] = xyx^{-1}y^{-1}$ ($x, y \in G$) is an L_2 -admissible binary operation on G . If H, K denote two arbitrary subgroups of G , then which respect to (2.4') the operation $[\cdot, \cdot]$ induces the following binary operation

$$(6.6) \quad [H, K] = \{[h, k]; h \in H, k \in K\}$$

on the complete lattice L_2 . It is easy to see that L_2 forms a complete groupoid-lattice under the „multiplication” (6.6) and under the lattice operations. The least element of L_2 is the unit element of G and its greatest element is G . It is remarkable that in general $[x, y] = [y, x]$ ($x, y \in G$) does not hold, but $[H, K] = [K, H]$ holds for all $H, K \in L_2$.

Example 6.3. Let $\mathfrak{F}_0 = \langle F_0; \cdot, 0 \rangle$ be a semigroup with 0. Similarly to Example 6.1 the set L_3 of all subsemigroups with zero of $\mathfrak{F}_0$ forms a complete groupoid-

lattice. It is interesting that the associative multiplication on F_0 induces by (6.5) an in general non-associative multiplication on L_3 . Namely if A, B, C , are sub-semigroups with zero of $\mathfrak{F}_0$, then for the elements $a_1, a_2 \in A, b_1 b_2 \in B$ and $c \in C$

$$d = (a_1 b_1 a_2 b_2) c \in (AB)C \text{ but in general } d \notin A(BC).$$

Example 6.4. Analogously to Example 6.1 the set L_4 of all subrings of a (not necessarily associative) ring R forms a complete groupoid-lattice, whose least element is 0 and the greatest one is R .

In the foregoing examples we considered only one (binary) operation of the given algebraic system and the set of all algebraic subsystems formed always a complete groupoid-lattice. In an other paper we want to write about complete groupoid-lattices, which satisfy some conditional associative and distributive laws. Now we give some illustrating examples for the results of §§ 2, 3 and 4.

Example 6.5. Let us consider again the set L_2 of all subgroups of a group G . If A, B are two subgroups of G , then the multiplication on G induces by (2.4') the multiplication $AB = \{ab; a \in A, b \in B\}$ in L_2 , satisfying the relation $AB = A \cup B$. The inverse-forming $a^{-1} (a \in G)$ is an L_2 -admissible unary operation in G , it induces by (2.4') the unary operation $A^{-1} = A(\in L_2)$ in L_2 . (It is evident that not the same identities hold for the multiplication and inverse-forming on G as for the induced operations on L_2 .) In view of Corollary 4.3 L_2 forms a compactly generated algebra-lattice under the mentioned two induced operations and under the set inclusion.

Example 6.6. Let $\mathfrak{Q} = \langle L; \wedge, \vee \rangle$ be a complete lattice with the least element 0 and with the greatest element e . Let $S_0(\mathfrak{Q})$ denote the set of all complete sublattices with zero of $\mathfrak{Q}$. If $A_\gamma (\gamma \in \Gamma)$ is a system of elements of $S_0(\mathfrak{Q})$ then in view of (2.4') the operations $\wedge$ and $\vee$ induce the following operations $\overline{\wedge}$ and $\overline{\vee}$ on $S_0(\mathfrak{Q})$:

$$\begin{aligned} \overline{\wedge} A_\gamma &= \left\{ \bigwedge_{\gamma \in \Gamma} x_\gamma; x_\gamma \in A_\gamma (\gamma \in \Gamma) \right\}, \\ \overline{\vee} A_\gamma &= \left\{ \bigvee_{\gamma \in \Gamma} x_\gamma; x_\gamma \in A_\gamma (\gamma \in \Gamma) \right\}. \end{aligned}$$

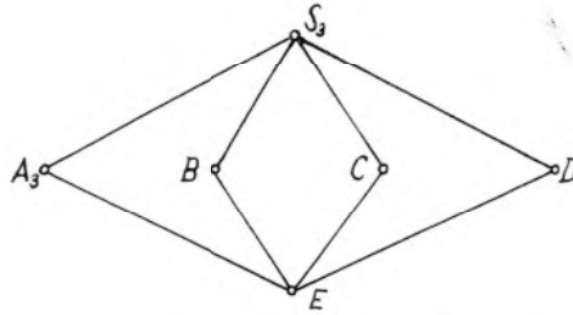
It is easy to show that $\overline{\wedge}$ and $\overline{\vee}$ are commutative and associative (infinitary) operations on $S_0(\mathfrak{Q})$. Naturally $S_0(\mathfrak{Q})$ forms a complete lattice under the set inclusion. If we denote by $\cap$ and $\cup$ the operations of this complete lattice, then $\langle S_0(\mathfrak{Q}); \overline{\wedge}, \overline{\vee}; \cap, \cup \rangle$ is a complete algebra-lattice.

Example 6.7. Let us consider the following operations $a+b, -a, ab, a \circ b = a+b-ab$ of an associative ring R ($a, b \in R$). If A, B are two subrings of R , then the given operations induce by (2.4') the following operations on the set L_5 of all subrings of R :

$$\begin{aligned} A+B &= \{a+b; a \in A; b \in B\}, \\ -A &= \{-a; a \in A\}, \\ AB &= \{ab; a \in A; b \in B\}, \\ A \circ B &= \{a \circ b; a \in A; b \in B\}. \end{aligned}$$

Naturally L_5 forms a complete lattice under the set inclusion. Let $\cap$ and $\cup$ denote the lattice operations. It is easy to show that the following relations hold ⁵⁾ $A + B = A \circ B = A \cup B$ and $-A = A$. Similarly to Example 6.3 the multiplication AB may not be associative.

Example 6.8. a) Let S_3 denote the symmetric permutation group of order 3. The subgroups of S_3 are: S_3 ; $A_3 = (1), (123), (132)$; $B = (1), (12)$; $C = (1), (13)$; $D = (1), (23)$; $E = (1)$. These subgroups form under the set inclusion a complete lattice H , whose diagram is:



In view of Example 6.2 H forms under the binary operation (6.6) a commutative groupoid, whose Caley-table is:

(6.8)

	E	B	C	D	A_3	S_3
E	E	E	E	E	E	E
B	E	E	A_3	A_3	A_3	A_3
C	E	A_3	E	A_3	A_3	A_3
D	E	A_3	A_3	E	A_3	A_3
A_3	E	A_3	A_3	A_3	E	A_3
S_3	E	A_3	A_3	A_3	A_3	A_3

Thus H is a complete groupoid-lattice and the operations on H are given by (6.7) and (6.8).

b) Conversely let $\mathfrak{B} = \langle V; \cdot; \wedge, \vee \rangle$ be a complete groupoid-lattice with the underlying set $V = 0, a, b, c, d, e$. It is assumed that $\mathfrak{B}$ is isomorphic to the complete groupoid-lattice H and the one-to-one mapping is given by

$$0 \rightarrow E, a \rightarrow A_3, b \rightarrow B, c \rightarrow C, d \rightarrow D, e \rightarrow S_3.$$

(The operations on V are defined according to (6.7) and (6.8).)

⁵⁾ Cf. Remark 1.

In view of Theorem 3.2 the algebra $\mathfrak{A} = \langle V; \cdot, \wedge, \vee, g_a, g_b, \dots \rangle$ ($a, b, \dots \in V$) belonging to $\mathfrak{B}$ has the property that the complete algebra-lattice of all subalgebras of $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

(We remark that $\mathfrak{B}$ is naturally a compactly generated algebra-lattice and $\mathfrak{A}$ is a finitary algebra.)

Example 6.9. Let N_0 denote the multiplicative semigroup of the non-negative integers. If we define on N_0 the following partial order $\equiv$:

$$a \equiv b \quad \text{if and only if} \quad b|a \quad (a, b \in N_0),$$

then N_0 forms under the relation $\equiv$ a complete lattice. Its greatest (least) element is the number 1 (zero). For any subset M of N_0 , $\sup M$ is the greatest common divisor of the elements of M . Moreover for finite M , $\inf M$ is the least common multiple of the elements of M and for infinite M , it is 0.

It is easy to see that the relations (6.1), (6.2) and (6.3) hold in N_0 , thus $\mathfrak{N}_0 = \langle N_0; \cdot; \sup, \inf \rangle$ is a complete groupoid-lattice. According to (3.1) one can define for every number $a (\in N_0)$ the following unary operation g_a on N_0 :

$$g_a(x) = \begin{cases} a, & \text{if } x|a, \\ 0, & \text{if } x \nmid a \end{cases} \quad (x \in N_0).$$

Let us consider the algebra $\mathfrak{A} = \langle N_0; \cdot; \sup, \inf, g_a, g_b, \dots \rangle$ ($a, b, \dots \in N_0$) belonging to $\mathfrak{N}_0$. From Lemma 3.1 it follows that every subalgebra of $\mathfrak{A}$ has the form $[0, a]$ ($a \in N_0$), where the interval $[0, a]$ consists of the multiples of the number a .

In view of Theorem 3.2 the set $S(\mathfrak{A})$ of all subalgebras of $\mathfrak{A}$ forms a complete groupoid-lattice isomorphic to $\mathfrak{N}_0$. The isomorphism is given by

$$a \rightarrow [0, a] \quad (a \in N_0).$$

References

- [1] G. BIRKHOFF and O. FRINK, Representations of lattices by sets, *Trans. Amer. Math. Soc.* **64** (1948), 299—316.
- [2] P. M. COHN, Universal algebra, *New York, Evanston, London, Tokio*, 1965.
- [3] L. FUCHS, On partially ordered algebras I, *Colloq. Math.* **14** (1966), 115—130.
- [4] L. FUCHS, On partially ordered algebras II, *Acta Sci. Math. Szeged*, **26** (1965), 35—41.
- [5] G. GRÄTZER, On the family of certain subalgebras of a universal algebra, *Indag. Math.* **27** (1965), 790—802.
- [6] G. GRÄTZER, Universal algebra (mimeographed).
- [7] G. GRÄTZER and E. T. SCHMIDT, Characterizations of congruence lattices of abstract algebras, *Acta Sci. Math. Szeged*, **24** (1963), 34—59.
- [8] H. M. MACNEILLE, Partially ordered sets, *Trans. Amer. Math. Soc.* **42** (1937), 416—460.
- [9] J. SŁOMINSKI, The theory of abstract algebras with infinitary operations, *Rozprawy Mat. (Warszawa)* **18** (1959), 1—67.
- [10] G. SZÁSZ, Introduction to lattice theory, *Budapest*, 1963.
- [11] M. WARD, The closure operators of a lattice, *Ann. of Math.* **43** (1942), 191—196.

(Received October 16, 1967)