

On the number of holomorphs of rings of order 8.

by

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Introduction. In a previous paper [1] we have determined the number of holomorphs of rings with additive group of type (p, p) and (p, p^2) , (p is a prime number). In this paper *) we want to discuss the rings with additive group of type $(2, 2, 2)$. As an abelian non-cyclic group of order 8 is either the direct sum of a cyclic group of order 2 and a cyclic group of order 4 or the direct sum of three cyclic groups of order 2, we get a survey of the number of holomorphs of all rings of order 8. Together with our previous results of [1] we have determined the number of holomorphs for all finite rings R with order less than 16. Our results for non-zero rings R with additive group R^+ of type $(2, 2, 2)$ can easily be generalized to the case of non-zero rings R with R^+ of type (p, p, p) , where p is a prime number. The zero-ring R with additive group R^+ of type $(2, 2, 2)$ has a large number of holomorphs, in fact 367. The question arises now to determine the number of non-isomorphic holomorphs for finite rings with a small number of elements, both for non-zero and zero-rings.

1. The non-zero rings R with additive group R^+ of type $(2, 2, 2)$

Let R be a ring, whose additive group $R^+ = (a_1) \oplus (a_2) \oplus (a_3)$ is the direct sum of three cyclic groups (a_1) , (a_2) and (a_3) . We assume, that R is not a zero-ring, i.e. the product ab ($a, b \in R$) does not vanish for all $a, b \in R$. The annihilator n_R of R is the set of all elements $a \in R$, such that $aR = Ra = 0$. R^2 is the ideal in R , generated by all products ab ($a, b \in R$). Both n_R and R^2 are characteristic subrings of R , which means that both n_R and R^2 are invariant under all double homothetisms of R (for the definitions and terminology we refer to our paper [1]).

As $O(R) = 8$, we have that the orders of n_R and R^2 are divisors of 8, i.e. 1, 2, 4 or 8. If $O(n_R) = 1$ or $n_R = (0)$, then R has one holomorph (WEINERT—EILHAUER [3]). If $O(n_R) = 8$ or $n_R = R$, then R is a zero-ring, which we have excluded. If $O(R^2) = 1$ or $R^2 = (0)$, then R is a zero-ring. If $O(R^2) = 8$ or $R^2 = R$, then R has one holomorph (VAN LEEUWEN [1]). Thus we have to investigate only the cases: a) $O(n_R) = 2$, $O(R^2) = 4$; b) $O(n_R) = 4$, $O(R^2) = 2$; c) $O(n_R) = 4$, $O(R^2) = 4$, and d) $O(n_R) = 2$, $O(R^2) = 2$.

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Case a) $O(n_R)=2$, $O(R^2)=4$.

$a_1)$ $n_R \cap R^2 = (0)$. In this case $R = n_R \oplus R^2$ is the ring-theoretic direct sum of its ideals n_R and R^2 and R has one holomorph (WEINERT—EILHAUER [3], Satz 4).

$a_2)$ $n_R \cap R^2 \neq (0)$. As $n_R \cap R^2 \subseteq n_R$, $O(n_R)=2$, we must have $O(n_R \cap R^2)=2$. Then $n_R \cap R^2 = n_R$ or $n_R \subseteq R^2$. Without loss of generality we may suppose that $n_R = \{0, a_1\}$ and $R^2 = \{0, a_1, a_2, a_1 + a_2\}$. One can construct 6 rings R with $n_R = \{0, a_1\}$ and $R^2 = \{0, a_1, a_2, a_1 + a_2\}$ and $R^+ = (a_1) \oplus (a_2) \oplus (a_3)$. These rings have the following multiplication tables:

A_1	a_2	a_3	A_2	a_2	a_3	A_3	a_2	a_3	A_4	a_2	a_3
a_2	0	a_1	a_2	0	a_1	a_2	a_2	0	a_2	a_2	a_2
a_3	a_1	a_2	a_3	a_1	$a_1 + a_2$	a_3	0	a_1	a_3	a_2	$a_1 + a_2$

A_5	a_2	a_3	A_6	a_2	a_3
a_2	$a_1 + a_2$	0	a_2	$a_1 + a_2$	$a_1 + a_2$
a_3	0	a_1	a_3	$a_1 + a_2$	a_2

Each of the rings A_i ($i=1, \dots, 6$) is commutative. By the condition $s_i(ab) = s_i(a)b$ for all $a, b \in A_i$, those endomorphisms s_i of A_i^+ are selected, which may occur as a first component of a double homothetism of A_i . These endomorphisms s_i form a subring $K_1(A_i)$ of $E(A_i^+)$, the endomorphism ring of A_i^+ . Likewise the endomorphisms s_k with $s_k(ab) = a(s_k b)$ for all $a, b \in A_i$ form a subring $K_2(A_i)$ of $E(A_i^+)$. As A_i is commutative, we get $K_1(A_i) = K_2(A_i) \subseteq E(A_i^+)$. For the uniqueness of the holomorph of a commutative ring A_i the commutativity of the ring $K_1(A_i) = K_2(A_i) \subseteq E(A_i^+)$ is necessary and sufficient ([3], Korollar, Satz 1). It is easy to check that for all rings A_i ($i=1, \dots, 6$) the ring $K_1(A_i)$ is commutative. Therefore each of the rings A_i has one holomorph. It may be remarked here that the ring A_1 , for instance, does not satisfy the conditions of a theorem of POLLÁK [2], which reads: If the ring R has a characteristic subring R' , which has one holomorph, and if each homomorphism of R/R' in n_R is the zero-homomorphism, then R has one holomorph. As A_1 has one holomorph, the conditions in this theorem are not necessary for the uniqueness of the holomorph. This is an example of a *finite* ring in which there is no proper characteristic subring R' satisfying Pollák's condition.

Case b) $O(n_R)=4$, $O(R^2)=2$.

$b_1)$ $n_R \cap R^2 = (0)$. In this case $R = n_R \oplus R^2$ is the direct sum of its ideals n_R and R^2 and as n_R has more than one holomorph (VAN LEEUWEN [1], Satz 3), R has more than one holomorph [1]. Now the subrings n_R and R^2 are characteristic subrings of R . Then the holomorphs of R exist in the form $H = H_1 \oplus H_2$, where H_1 is an arbitrary holomorph of n_R and H_2 is an arbitrary holomorph of R^2 , (POLLÁK [2]). The zero-ring n_R with $n_R^+ = (a_1) \oplus (a_2)$, $O(a_1) = O(a_2) = 2$, has $2^2 + 2 + 3 = 9$ holomorphs (VAN LEEUWEN [1], Satz 3). The ring $R^2 = (a_3)$ has one holomorph. Therefore $R = n_R \oplus R^2$ has 9 holomorphs. All rings R with $O(n_R)=4$, $O(R^2)=2$, $R^+ = (a_1) \oplus (a_2) \oplus (a_3)$, $O(a_1) = O(a_2) = O(a_3) = 2$, are isomorphic. Hence all of these rings have 9 holomorphs.

$b_2) n_R \cap R^2 \neq (0)$. As $n_R \cap R^2 \subseteq R^2$, $O(R^2)=2$, we must have $O(n_R \cap R^2)=2$. Then $n_R \cap R^2 = R^2$ or $R^2 \subseteq n_R$. Without loss of generality we may suppose that $R^2 = \{0, a_1\}$ and $n_R = \{0, a_1, a_2, a_1 + a_2\}$. Then $a_1^2 = a_1 a_2 = a_1 a_3 = a_2 a_1 = a_2^2 = a_2 a_3 = a_3 a_1 = a_3 a_2 = 0$. And $a_3^2 = a_1$, as $a_3^2 = 0$ implies that R is a zero-ring. For this multiplication the ring R is a non-zero ring, which is commutative and has 9 maximal rings of related double homothetisms and therefore 9 holomorphs. Therefore all of the rings in this case $b_2)$ have 9 holomorphs.

Case c) $O(n_R)=4$, $O(R^2)=4$.

Suppose $n_R = \{0, a, b, a+b\}$ and $c \in R$ with $c \notin n_R$. Then $c+a, c+b, c+a+b$ belong to R , but none of them belongs to n_R and $R = \{0, a, b, c, a+b, a+c, b+c, a+b+c\}$. As $aR = Ra = 0$ and $bR = Rb = 0$ we get $R^2 = \{0, c^2\}$, which is a contradiction to $O(R^2)=4$. Thus there are no rings possible in this case.

Case d) $O(n_R)=2$, $O(R^2)=2$.

$d_1) n_R \cap R^2 = (0)$. Suppose $n_R = \{0, a\}$ and $R^2 = \{0, b\}$ with $a \neq b$. Again, if $c \neq a$, $c \neq b$ ($c \in R$), then $R = \{0, a, b, c, a+b, a+c, b+c, a+b+c\}$. If $b^2 = 0$, then from $bc = b$ we would get $(bc)c = bc = b = b(c^2)$, or $c^2 = b$, but then $b^2 = b$. Contradiction. Thus $b^2 = 0$ implies $bc = 0$. Similarly $b^2 = 0$ implies $cb = 0$. But now $bR = Rb = 0$ or $b \in n_R$, which is impossible. We conclude: $b^2 = b$.

If $bc = b$, then $(bc)b = b^2 = b = b(cb)$, and $cb = b$. Also $(bc)c = bc = b = b(c^2)$, and $c^2 = b$. Then $(b+c)R = R(b+c) = 0$, which implies $b+c \in n_R$. Contradiction. If $bc = 0$, then $(bc)b = 0 = b(cb)$, or $cb = 0$. Also $(bc)c = 0 = b(c^2)$ and $c^2 = 0$. Then $cR = Rc = 0$ or $c \in n_R$. Contradiction. Thus there is no ring satisfying the conditions of this case.

$d_2) n_R \cap R^2 \neq (0)$. It follows now that $n_R = R^2$. Without loss of generality we may suppose that $n_R = R^2 = \{0, a_1\}$. For each of the elements $a_2^2, a_2 a_3, a_3 a_2$ and a_3^2 ($\in R$) one can choose either 0 or a_1 , but not $a_2^2 = a_2 a_3 = a_3 a_2 = a_3^2 = 0$, as R is not a zero-ring. One can construct 12 non-zero rings R with $n_R = R^2 = \{0, a_1\}$ and $R^+ = (a_1) \oplus (a_2) \oplus (a_3)$. These rings have the following multiplication tables:

$B_1:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & 0 & a_1 \\ \hline a_3 & a_1 & 0 \\ \hline \end{array}$	$B_2:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & 0 & a_1 \\ \hline a_3 & a_1 & a_1 \\ \hline \end{array}$	$B_3:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & a_1 & 0 \\ \hline a_3 & 0 & a_1 \\ \hline \end{array}$	$B_4:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & a_1 & 0 \\ \hline a_3 & a_1 & a_1 \\ \hline \end{array}$
$B_5:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & a_1 & a_1 \\ \hline a_3 & 0 & a_1 \\ \hline \end{array}$	$B_6:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & a_1 & a_1 \\ \hline a_3 & a_1 & 0 \\ \hline \end{array}$	$B_7:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & 0 & 0 \\ \hline a_3 & a_1 & 0 \\ \hline \end{array}$	$B_8:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & 0 & 0 \\ \hline a_3 & a_1 & a_1 \\ \hline \end{array}$
$B_9:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & 0 & a_1 \\ \hline a_3 & 0 & 0 \\ \hline \end{array}$	$B_{10}:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & 0 & a_1 \\ \hline a_3 & 0 & a_1 \\ \hline \end{array}$	$B_{11}:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & a_1 & 0 \\ \hline a_3 & a_1 & 0 \\ \hline \end{array}$	$B_{12}:$	$\begin{array}{ c c c } \hline a_2 & a_3 & \\ \hline a_2 & a_1 & a_1 \\ \hline a_3 & 0 & 0 \\ \hline \end{array}$

As the sets $\{B_2, B_3, B_6\}$, $\{B_4, B_5\}$, $\{B_7, B_8, B_9, B_{10}, B_{11}, B_{12}\}$ are consisting of

isomorphic rings each, we need only to consider the rings B_1, B_2, B_4 and B_7 . The ring B_1 is commutative. Therefore $K_1(B_1) = K_2(B_1)$ (see case a_2). As the ring $K_1(B_1)$ is commutative, the ring B_1 has one holomorph ([3], Korollar Satz 1). For the same reason, the ring B_2 has one holomorph.

Now we consider the ring B_4 . Let s_i be the first component of a double homothetism of B_4 . Then $s_i(a_2^2) = s_i(a_1) = s_i(a_2)a_2$, and $s_i(a_1) = 0$ or a_1 . From $s_i(a_2a_3) = 0 = s_i(a_2)a_3$ it follows that $s_i(a_2) = 0, a_1, a_2$ or $a_1 + a_2$. From $s_i(a_3a_2) = s_i(a_1) = s_i(a_3)a_2$ and $s_i(a_3^2) = s_i(a_1) = s_i(a_3)a_3$ we infer that $s_i(a_3) = 0, a_1, a_3$ or $a_1 + a_3$. It turns out that $s_i(a_1) = 0$ implies $s_i(a_2) = 0$ or a_1 and $s_i(a_3) = 0$ or a_1 and that $s_i(a_1) = a_1$ implies $s_i(a_2) = a_2$ or $a_1 + a_2$ and $s_i(a_3) = a_3$ or $a_1 + a_3$. Let s_k be the second component of a double homothetism of B_4 . Then likewise we find that $s_k(a_1) = 0$ implies $s_k(a_2) = 0$ or a_1 and $s_k(a_3) = 0$ or a_1 and that $s_k(a_1) = a_1$ implies $s_k(a_2) = a_2$ or $a_1 + a_2$ and $s_k(a_3) = a_3$ or $a_1 + a_3$. Thus $K_1(B_4) = K_2(B_4)$. The ring $K_1(B_4)$ consists of the same endomorphisms of $(a_1) \oplus (a_2) \oplus (a_3)$ as the ring $K_1(B_2)$ or $K_1(B_1)$. Therefore $K_1(B_4)$ is commutative. The condition $s_k(s_i a) = s_i(s_k a)$ is satisfied for all $s_i \in K_1(B_4)$, $s_k \in K_2(B_4)$ and for all $a \in B_4$. The condition $(s_k a)b = a(s_i b)$ for all $a, b \in B_4$ with $s_i \in K_1(B_4)$ and $s_k \in K_2(B_4)$ implies that we cannot use all possible pairs (s_i, s_k) ($s_i \in K_1(B_4)$, $s_k \in K_2(B_4)$) as double homothetisms of B_4 . But if (s_i, s_k) and (s'_i, s'_k) are double homothetisms of B_4 , then $s_i s'_k = s'_k s_i$ and $s_k s'_i = s'_i s_k$, as $K_1(B_4) = K_2(B_4)$ is commutative. All double homothetisms of B_4 are pairwise related, therefore B_4 has one holomorph.

Finally we investigate the ring B_7 . From the conditions $s_i(ab) = s_i(a)b$, $s_k(ab) = as_k(b)$, $as_i(b) = s_k(a)b$ for s_i, s_k endomorphisms of B_7^+ and all $a, b \in B_7$ we get that $s_i(a_1) = 0$ or a_1 , $s_i(a_2) = 0, a_1, a_2$ or $a_1 + a_2$, $s_i(a_3) = 0, a_1, a_3$ or $a_1 + a_3$ and $s_k(a_1) = 0$ or a_1 , $s_k(a_2) = 0, a_1, a_2$ or $a_1 + a_2$, $s_k(a_3) = 0, a_1, a_3$ or $a_1 + a_3$. Moreover if $s_i(a_1) = 0$ then $s_i(a_3) = 0$ or a_1 and if $s_i(a_1) = a_1$ then $s_i(a_3) = a_3$ or $a_1 + a_3$; likewise if $s_k(a_1) = 0$ then $s_k(a_2) = 0$ or a_1 and if $s_k(a_1) = a_1$ then $s_k(a_2) = a_2$ or $a_1 + a_2$. Also if $s_i(a_2) = 0$ or a_1 then $s_k(a_3) = 0$ or a_1 and if $s_i(a_2) = a_2$ or $a_1 + a_2$ then $s_k(a_3) = a_3$ or $a_1 + a_3$. A pair of endomorphisms (s_i, s_k) of B_7^+ satisfying the above conditions is a double homothetism of B_7 if $s_k(s_i a) = s_i(s_k a)$ for all $a \in B_7$. It turns out, that the set of all double homothetisms (s_i, s_k) of B_7 consists of 72 elements. The double homothetisms (s_i, s_k) and (s'_i, s'_k) of B_7 are related if $s_i s'_k = s'_k s_i$ and $s_k s'_i = s'_i s_k$ hold. Each set of pairwise related double homothetisms of B_7 is contained in a maximal ring D of this kind. The ring B_7 has 4 maximal rings of related double homothetisms and therefore 4 holomorphs.

2. The zero-ring R with additive group R^+ of type $(2, 2, 2)$

In this section R will be a zero-ring i.e. $ab = 0$ for all $a, b \in R$ and $R^+ = (a_1) \oplus (a_2) \oplus (a_3)$ is the direct sum of 3 cyclic groups, each of order 2. An endomorphism s of R^+ is determined by the images $s(a_1)$, $s(a_2)$ and $s(a_3)$. We start with the zero-endomorphism: $s(a_1) = 0, s(a_2) = 0, s(a_3) = 0$ and call it s_1 . Then $s_2(a_1) = 0, s_2(a_2) = 0, s_2(a_3) = a_1$; $s_3(a_1) = 0, s_3(a_2) = 0, s_3(a_3) = a_2$, etc. The endomorphisms of R^+ are indexed „lexicographically” and are in this order $s_1, \dots, s_{512}$, where $s_{512}(a_1) = a_1 + a_2 + a_3, s_{512}(a_2) = a_1 + a_2 + a_3, s_{512}(a_3) = a_1 + a_2 + a_3$. For the zero-ring R a pair of endomorphisms (s_i, s_k) of R is a double homothetism of R if $s_i s_k = s_k s_i$. Two double homothetisms (s_i, s_k) and (s'_i, s'_k) are related if $s_i s'_k = s'_k s_i$ and $s_k s'_i = s'_i s_k$.

Our problem is to determine the number of holomorphs of R , i.e. the number of maximal rings of related double homothetisms of R . In order to do so, we introduce the following notation for the set of endomorphisms of R^+ , consisting of elements which are commuting with a given s : $\{s\} = \{s_i \in E(R^+), s_i s = s s_i\}$. There are 2 endomorphisms s in $E(R^+)$, such that $\{s\}$ has order 512, namely s_1 and s_{84} : $a_1 \rightarrow a_1$, $a_2 \rightarrow a_2$, $a_3 \rightarrow a_3$. There are 98 endomorphisms s in $E(R^+)$, such that $\{s\}$ has order 32. These 98 endomorphisms can be divided into pairs, such that each pair consists of two endomorphisms s_b, s_a with the same elements in $\{s_a\}$ and $\{s_b\}$, $s_a \neq s_b$. The remaining 412 elements of $E(R^+)$ are each commuting with 8 endomorphisms of $E(R^+)$.

A. First we consider the sets $\{s\}$ with order 8. If $\{s\}$ is a set of order 8, and $s_a, s_b \in \{s\}$, then $s_a s_b = s_b s_a$. Now we distinguish two cases:

- (i) if s_c is an arbitrary element of $\{s\}$, $s_c \neq s_1, s_c \neq s_{84}$, then $\{s_c\} = \{s\}$.
- (ii) there is at least one element $s_v \in \{s\}$, $s_v \neq s_1, s_v \neq s_{84}$, such that $\{s_v\} \neq \{s\}$. In this case $\{s\} \subset \{s_v\}$ and $\{s_v\}$ has order 32.

Case (i). Let $\{s\}$ be a set of endomorphisms of $E(R^+)$ of order 8. If s_c, s_b are arbitrary in $\{s\}$, then (s_a, s_b) is a double homothetism of R , as $s_a s_b = s_b s_a$. It is clear that every pair of double homothetisms $(s_a, s_b), (s'_a, s'_b)$ is related. The set of all double homothetisms, obtained in this way, is a maximal set and therefore a maximal ring of related double homothetisms of R . For let (s_t, s_u) be related to (s_a, s_b) with $s_a, s_b \in \{s\}$. Then $s_t s_b = s_b s_t$ or $s_t \in \{s_b\} = \{s\}$ and $s_a s_u = s_u s_a$ or $s_u \in \{s_a\} = \{s\}$. Thus we get 8 maximal rings of related double homothetisms of R and therefore 8 holomorphs of R . These holomorphs are denoted by P_i ($i = 1, \dots, 8$).

Case (ii). Let s be an endomorphism which commutes with 8 endomorphisms, so that $\{s\}$ has order 8 and let $s_v \in \{s\}$ with $\{s\} \subset \{s_v\}$, where $\{s_v\}$ has order 32. If s_a and s_b are arbitrary in $\{s\}$, then (s_a, s_b) is a double homothetism of R , as $s_a s_b = s_b s_a$. Again we form the set of all pairs (s_a, s_b) with $s_a, s_b \in \{s\}$. Then we obtain a set S of related double homothetisms of R , since, if $s_a, s_b, s'_a, s'_b \in \{s\}$, then $s_a s'_b = s'_b s_a$ and $s_b s'_a = s'_a s_b$, whence the double homothetisms (s_a, s_b) and (s'_a, s'_b) are related. We state that S is a maximal set of related double homothetisms of R . Indeed, let (s_t, s_u) be a double homothetism related to all double homothetisms of S . In particular, (s_t, s_u) is related to (s, s) which means $s_t s = s s_t$ and $s_u s = s s_u$ or $s_t \in \{s\}$ and $s_u \in \{s\}$. Therefore $(s_t, s_u) \in S$, hence S is maximal with respect to the property of relatedness. In this way we get 133 maximal rings S_i of related double homothetisms of R and therefore 133 holomorphs of R , which we denote by Q_i ($i = 1, \dots, 133$).

B. Next we consider the sets $\{s\}$ with order 32. As we have remarked, there are 49 different sets of this kind. Let $\{s_c\}$ and $\{s_d\}$ be two *different* sets, both of order 32, and such that $s_c s_d = s_d s_c$. It may be remarked here, that two arbitrary elements s_a, s_b of a given set $\{s\}$ of order 32 need not be commuting. Now (s_c, s_d) is a double homothetism of R . Then we consider the endomorphisms of $E(R^+)$ which belong to the set $\mathcal{A} = \{s_c\} \cap \{s_d\}$. The endomorphisms of $\mathcal{A}$ are commuting and $\mathcal{A}$ has order 8. There are now two cases: (a) there is at least one endomorphism s in $\mathcal{A}$, such that $\{s\}$ has order 8, then $\mathcal{A} = \{s\}$; (b) for all endomorphisms s in $\mathcal{A}$, except s_1 and s_{84} , the set $\{s\}$ has order 32. In both cases we form the set of all pairs (s_k, s_l) with $s_k, s_l \in \mathcal{A}$. In case (a) we get a set S of pairwise related double homothetisms which were obtained already in case (ii) of **A**. In case (b) the elements of $\mathcal{A} = \{s_c\} \cap \{s_d\}$ are: $s_1, s_{84}, s_c, s_d, s_e$ such that $\{s_e\} = \{s_c\}$, s_f such that $\{s_f\} =$

$= \{s_d\}$, s_g with $\{s_g\} \cap \{s_c\} = \{s_g\} \cap \{s_d\} = \mathcal{A}$ and s_h with $\{s_h\} = \{s_g\}$. The set of all pairs (s_k, s_l) with $s_k, s_l \in \mathcal{A}$ is again a set S' of pairwise related double homothetisms of R . Also S' is a maximal set of related double homothetisms of R . For, let (s_t, s_u) be a double homothetism related to all double homothetisms of S' . Then (s_t, s_u) is related to (s_c, s_c) or $s_t s_c = s_c s_t$ and $s_u s_c = s_c s_u$. Likewise we have $s_t s_d = s_d s_t$ and $s_u s_d = s_d s_u$. Hence $s_t \in \{s_c\} \cap \{s_d\} = \mathcal{A}$, and $s_u \in \mathcal{A}$, whence $(s_t, s_u) \in S'$. Thus we get 42 maximal rings S'_i of related double homothetisms of R and therefore 42 holomorphs of R , which we denote by R_i ($i = 1, \dots, 42$).

C. Let $\{s_c\}$ and $\{s_d\}$ be *equal* sets, both of order 32, with $s_c \neq s_d$. As s_c and s_d belong to the same set $\{s_c\} = \{s_d\}$, it is clear that $s_c s_d = s_d s_c$. Now we can form the set S'' consisting of all pairs $(s_1, s_t), (s_{84}, s_u), (s_c, s_x), (s_d, s_y)$, where s_t, s_u, s_x and s_y run through the set $\{s_c\}$. Then S'' consists of related double homothetisms of R . Let (s_e, s_f) be a double homothetism related to all double homothetisms of S'' . We are going to prove that $(s_e, s_f) \in S''$. In $\{s_c\}$ we can find two endomorphisms s_a, s_b such that both $\{s_a\}$ and $\{s_b\}$ have order 8. We show that $\{s_a\} \cap \{s_b\} = (s_1, s_{84}, s_c, s_d)$. Indeed, as $s_a \in \{s_c\} = \{s_d\}$ we have that $s_c \in \{s_a\}$ and $s_d \in \{s_a\}$, $s_b \in \{s_c\} = \{s_d\}$ implies that $s_c \in \{s_b\}$ and $s_d \in \{s_b\}$. Hence $(s_1, s_{84}, s_c, s_d) \subseteq \{s_a\} \cap \{s_b\}$ or the order of $\{s_a\} \cap \{s_b\}$ is ≥ 4 . But as the orders of $\{s_a\}$ and $\{s_b\}$ are 8, and $\{s_a\} \neq \{s_b\}$, we must have that the order of $\{s_a\} \cap \{s_b\}$ is ≤ 4 . This proves that $\{s_a\} \cap \{s_b\} = (s_1, s_{84}, s_c, s_d)$. From (s_e, s_f) is related to (s_c, s_a) we infer that $s_e s_a = s_a s_e$ or $s_e \in \{s_a\}$. Likewise $s_e s_b = s_b s_e$ or $s_e \in \{s_b\}$. Hence $s_e \in \{s_a\} \cap \{s_b\} = (s_1, s_{84}, s_c, s_d)$. From (s_e, s_f) is related to (s_c, s_a) we also infer that $s_f s_c = s_c s_f$ or $s_f \in \{s_c\}$. Thus $(s_e, s_f) \in S''$. It follows that S'' is a maximal ring of related double homothetisms of R . As we have 49 different pairs s_c, s_d with $\{s_c\} = \{s_d\}$ and $O(\{s_c\}) = 32$, we can form 49 maximal rings S''_i and we get 49 holomorphs of R .

In the same way we can form the set S''' consisting of the pairs $(s_t, s_1), (s_u, s_{84}), (s_x, s_c), (s_y, s_d)$, where s_t, s_u, s_x, s_y run through the set $\{s_c\} = \{s_d\}$ ($s_c \neq s_d$, $O(\{s_c\}) = 32$). As above, we can prove that S''' is a maximal ring of related double homothetisms of R . Thus we get again 49 holomorphs of R , which yields altogether 98 holomorphs of R , denoted by T_i ($i = 1, \dots, 98$).

D. Let $\{s_c\}$ and $\{s_d\}$ be two *different* sets, both of order 32, and such that $s_c s_d \neq s_d s_c$. Then we consider the endomorphisms of $E(R^+)$, which belong to the set $\mathcal{A}' = \{s_c\} \cap \{s_d\}$. The order of $\mathcal{A}'$ is 4 or 8.

If $\mathcal{A}'$ has order 4, then the elements of $\mathcal{A}'$ are (s_1, s_{84}, s_k, s'_k) , where $s_k \in \{s_c\} \cap \{s_d\}$ and $\{s'_k\} = \{s_k\}$ with $s_k \neq s'_k$. Now we can form the set of all pairs $(s_1, s_t), (s_{84}, s_u), (s_k, s_x), (s'_k, s_y)$, where s_t, s_u, s_x and s_y run through the set $\{s_k\}$; we get a maximal ring of related double homothetisms of R , which was obtained already in case **C**. The set of all pairs $(s_t, s_1), (s_u, s_{84}), (s_x, s_k)$ and (s_y, s'_k) is a maximal ring of related double homothetisms of R , if s_t, s_u, s_x and s_y run through the set $\{s_k\}$. This ring, too, is one of the rings of case **C**. If $\mathcal{A}'$ has order 8, then the elements of $\mathcal{A}'$ are $(s_1, s_{84}, s_e, s'_e, s_f, s'_f, s_g, s'_g)$, where each of the last 6 endomorphisms commutes with 32 endomorphisms of $E(R^+)$. The notation s_e, s'_e indicates that the sets $\{s_e\}$ and $\{s'_e\}$ are equal, and likewise for s_f and s'_f , and s_g and s'_g . In $E(R^+)$ we can find an endomorphism, say, $s_h, s_h \neq s_c, s_h \neq s_d, s_h \neq s'_c, s_h \neq s'_d$, such that $\{s_c\} \cap \{s_d\} = \{s_h\} \cap \{s'_h\} = \{s_c\} \cap \{s_h\}$. Now we form the set $\mathcal{B}' : (s_1, s_{84}, s_c, s'_c, s_d, s'_d, s_h, s'_h)$. As $\mathcal{A}' \subset \{s_h\} = \{s'_h\}$, it follows that each element of $\mathcal{B}'$ is commuting with each element of $\mathcal{A}'$. The set (s_k, s_l) of pairs $s_k \in \mathcal{B}', s_l \in \mathcal{A}'$ is a set $S^\wedge$ of related double homothetisms

of R . Moreover $S^\wedge$ is a maximal set of related double homothetisms of R . For, let (s_t, s_u) be a double homothetism related to all double homothetisms of $S^\wedge$. Then $s_us_c = s_cs_u$ and $s_us_d = s_ds_u$, hence $s_u \in \{s_c\} \cap \{s_d\} = \mathcal{A}'$. As $s_ts_e = s_es_t$ and $s_ts_f = s_fs_t$, it follows that $s_t \in \{s_e\} \cap \{s_f\}$. But $\{s_e\} \cap \{s_f\} = \mathcal{B}'$, whence $s_t \in \mathcal{B}'$. Therefore $(s_t, s_u) \in S^\wedge$. The set of all pairs (s_l, s_k) with $s_l \in \mathcal{A}'$, $s_k \in \mathcal{B}'$ is likewise a maximal set $S^{\wedge\wedge}$ of related double homothetisms of R . In this way we obtain 42 pairs $S^\wedge, S^{\wedge\wedge}$ of maximal sets, hence 84 maximal rings of related double homothetisms of R . The 84 corresponding holomorphs of R are denoted by U_i ($i = 1, \dots, 84$).

E. Finally we consider the two sets $\{s_1\}$ and $\{s_{84}\}$, both consisting of all endomorphisms of R^+ . Here we form the set S^{iv} of all pairs (s_1, s_k) and (s_{84}, s_n) where s_k and s_n run through $E(R^+)$. S^{iv} consists of related double homothetisms of R and it is a maximal set of this kind.

Likewise the set S^v , consisting of all pairs (s_k, s_1) and (s_n, s_{84}) , where s_k, s_n run through $E(R^+)$, is a maximal ring of related double homothetisms of R . Thus we obtain 2 holomorphs of R , denoted by V_1, V_2 . For a zero-ring R , whose additive group $R^+ = (a_1) \oplus (a_2) \oplus (a_3)$ is the direct sum of three cyclic groups each of order 2, we have determined 367 holomorphs. We prove now that these are *all* holomorphs of R .

Let U be an arbitrary maximal ring of related double homothetisms of R .

a. Suppose there is an element $(s_i, s_k) \in U$, $s_i \neq s_1, s_{84}, s_k \neq s_1, s_{84}$, such that $\{s_i\}$ has order 8, and if s_c is arbitrary in $\{s_i\}$, then $\{s_c\} = \{s_i\}$ ($s_c \neq s_1, s_{84}$). As we have seen in case (i) of **A**, the set of all pairs (s_a, s_b) with $s_a, s_b \in \{s_i\}$ is a maximal ring M of related double homothetisms of R . It follows that $(s_i, s_k) \in U \cap M$. Now let (s_c, s_d) be an arbitrary element of U ($s_c \neq s_1, s_{84}, s_d \neq s_1, s_{84}$). As $s_cs_k = s_ks_c$ and $s_k \in \{s_i\}$, we get that $\{s_k\} = \{s_i\}$ and $s_c \in \{s_i\}$. From $s_ds_i = s_is_d$ it follows that $s_d \in \{s_i\}$. Therefore $(s_c, s_d) \in M$. The elements (s_c, s_d) in U with either s_c or s_d or both equal to s_1 or s_{84} belong to M , as $s_1, s_{84} \in \{s_i\}$. Thus $U \subseteq M$, and as U is maximal, $U = M$. Then the holomorph corresponding to U is one of the holomorphs P_i ($i = 1, \dots, 8$). In the sequel we need the following properties of the ring $E(R^+)$:

Property I. If $\{s_a\}$ has order 8, $\{s\}$ has order 32 and $s_a \in \{s\}$, then $\{s_a\} \subset \{s\}$

Property II. If $s_i, s_k \in \{s_i\} \cap \{s_k\}$ and both $\{s_i\}$ and $\{s_k\}$ have order 8, then $\{s_i\} = \{s_k\}$.

b. Suppose there is an element $(s_i, s_k) \in U$, $s_i \neq s_1, s_{84}, s_k \neq s_1, s_{84}$ with $s_i, s_k \in \{s\}$, $\{s\}$ being a set of order 8 of case (ii) of **A**. Moreover we assume that at least one of the two sets $\{s_i\}$ and $\{s_k\}$ is of order 8, say $\{s_i\}$. As $s_is = ss_i$ and both $\{s\}$ and $\{s_i\}$ have order 8, we get $\{s\} = \{s_i\}$ (property II).

b₁. $\{s_k\}$ has order 8, which implies that $\{s\} = \{s_i\} = \{s_k\}$. Let (s_a, s_b) be an arbitrary element of U . From (s_a, s_b) is related to (s_i, s_k) it follows that $s_a \in \{s_k\} = \{s_i\}$ and $s_b \in \{s_i\}$. This means that $U \subseteq S_i$, where S_i is a maximal ring of related double homothetisms of R , obtained in case (ii) of **A**. From the maximality of U we infer that $U = S_i$, hence the holomorph corresponding to U is one of the holomorphs Q_i ($i = 1, \dots, 133$).

b₂. $\{s_k\}$ has order 32, which implies $\{s_i\} \subset \{s_k\}$ (property I).

b₂(i). All double homothetisms of U have as second component s_1, s_{84}, s_k , or s'_k , where s'_k is the uniquely determined endomorphism of R^+ with $\{s'_k\} = \{s_k\}$, $s_k \neq s'_k$. Let (s_a, s_b) be an arbitrary element of U . Then $s_a \in \{s_k\}$, hence $U \subseteq S_i'''$, where S_i''' is one of the maximal rings of related double homothetisms of R , obtained in **C**.

Again $U = S_i'''$ and the holomorph corresponding to U is one of the holomorphs T_i ($i = 1, \dots, 98$).

b_2 .(ii). There is at least one double homothetism in U , say (s_c, s_d) , such that s_d is not one of the endomorphisms s_1, s_{84}, s_k, s'_k of case b_2 (i). As (s_c, s_d) is related to (s_i, s_k) it follows that $s_d \in \{s_i\} \subset \{s_k\}$. Let (s_a, s_b) be an arbitrary element of U . Then (s_a, s_b) is related both to (s_i, s_k) and (s_c, s_d) . Therefore $s_a \in \{s_k\} \cap \{s_d\}$.

b_2 (iia). Suppose $\{s_d\}$ has order 8, then $\{s_d\} \subset \{s_k\}$ (property I) and $s_a \in \{s_d\}$. From $s_d s_i = s_i s_d$ and both $\{s_d\}$ and $\{s_i\}$ have order 8, it follows that $\{s_d\} = \{s_i\}$ (property II). Hence $s_a \in \{s_i\}$. From (s_a, s_b) is related to (s_i, s_k) we infer that $s_b \in \{s_i\}$. Thus in this case $U \subseteq S_i$, where S_i is one of the maximal rings of related double homothetisms of R , obtained in case (ii) of **A**. Then $U = S_i$ and the holomorph corresponding to U is one of the holomorphs Q_i ($i = 1, \dots, 133$).

b_2 (iib). Suppose $\{s_d\}$ has order 32. Now $s_k s_d = s_d s_k$, both $\{s_k\}$ and $\{s_d\}$ have order 32 and $\{s_k\} \neq \{s_d\}$. From (s_a, s_b) is related to (s_i, s_k) it follows that $s_a \in \{s_d\} \cap \{s_k\}$. Also $s_d \in \{s_i\}$ and, as $\{s_i\}$ has order 8, one gets $s_d s_b = s_b s_d$ or $s_b \in \{s_d\}$. Hence $s_b \in \{s_k\} \cap \{s_d\}$. Now $U \subseteq \mathcal{A}_i$, where $\mathcal{A}_i$ is one of the maximal rings of related double homothetisms of R of case **B**. Then $U = \mathcal{A}_i$ and the holomorph corresponding to U is one of the holomorphs R_i ($i = 1, \dots, 42$).

Remark. The case: "there exists an element $(s_i, s_k) \in U$, $s_i \neq s_1, s_{84}$, $s_k \neq s_1, s_{84}$ with $s_i, s_k \in \{s\}$, $\{s\}$ being a set of order 8 of case (ii) of **A** and $\{s_k\}$ having order 8" is quite similar to the case b just finished.

c . Suppose there is an element $(s_i, s_k) \in U$ ($s_i \neq s_1, s_{84}$; $s_k \neq s_1, s_{84}$) with $s_i, s_k \in \{s\}$, where $\{s\}$ is a set of order 8 of case (ii) of **A**. Now we assume that both $\{s_i\}$ and $\{s_k\}$ have order 32.

First we remark that we may assume that for every element $(s_a, s_b) \in U$ ($s_a \neq s_1, s_{84}$; $s_b \neq s_1, s_{84}$) both $\{s_a\}$ and $\{s_b\}$ have order 32. For let $(s_c, s_d) \in U$, where, say s_c , has order 8. As $s_d \in \{s_c\}$, we have that $s_c, s_d \in \{s_c\}$, $\{s_c\}$ being a set of order 8. Then we get a ring U , already considered in case a or case b . Secondly we state that if $(s_c, s_t) \in U$, $s_c \neq s_1, s_{84}$, then $(s'_c, s_t) \in U$, where s'_c is the uniquely determined element of $E(R^+)$ with $\{s_c\} = \{s'_c\}$, $s_c \neq s'_c$. For, if (s_a, s_b) is an arbitrary element of U , then $s'_c s_b = s_b s'_c$, as $s_c s_b = s_b s_c$ and $s_a s_t = s_t s_a$. Likewise from $(s_c, s_t) \in U$ it follows that $(s_c, s'_t) \in U$. In the sequel the notation s_a, s'_a means that the sets $\{s_a\}$ and $\{s'_a\}$ are equal, both of order 32, and $s_a \neq s'_a$.

c_1 . All double homothetisms of U have as a first component s_1, s_{84}, s_i or s'_i . Let (s_a, s_b) be an arbitrary element of U . Then $s_b \in \{s_i\}$ and $U \subseteq S_i''$, where S_i'' is one of the maximal rings of related double homothetisms of R of case **C**. The maximality of U implies $U = S_i''$. The holomorph corresponding to U is one of the holomorphs T_i ($i = 1, \dots, 98$).

c_2 . There is at least one double homothetism of U , say (s_c, s_d) , such that $s_c \notin (s_1, s_{84}, s_i, s'_i)$. Now the endomorphisms $s_1, s_{84}, s_i, s'_i, s_c, s'_c$ occur as first components of double homothetisms of U .

c_2 (i). First we suppose that $s_i s_c = s_c s_i$. Let (s_a, s_b) be an arbitrary element of U , then $s_b \in \{s_c\} \cap \{s_i\}$. Therefore the set K of all second components of double homothetisms of U is a subset of a set $\mathcal{A}_i = \{s_c\} \cap \{s_i\}$ of case **B**. Now $s_k, s'_k \in K$, $s_k \neq s_1, s_{84}$, hence the order of $K \geq 4$. If the order of K is 4, then all double homothetisms of U have as second component s_1, s_{84}, s_k or s'_k . This case has been considered in c_1 , hence $O(K) > 4$. Then the endomorphisms $s_1, s_{84}, s_k, s'_k, s_i, s'_i$ occur in K , where $s_i \neq s_1, s_{84}, s_k, s'_k$ and $s_i \in \{s_c\} \cap \{s_i\}$. As s_a commutes with all second components,

it follows that $s_a \in \{s_k\} \cap \{s_l\} = \mathcal{A}_i$. Thus in this case $U \subseteq S'_i$, where S'_i is a maximal ring of related double homothetisms of R of case **B**. As U is also a maximal ring, we get that $U = S'_i$. The holomorph corresponding to U is one of the holomorphs R_i ($i = 1, \dots, 42$).

$c_2(ii)$. Secondly we suppose that $s_c s_i \neq s_i s_c$. Again, if (s_a, s_b) is an arbitrary element of U , $s_b \in \{s_c\} \cap \{s_i\}$. Therefore the set L of all second components of double homothetisms of U is a subset of a set $\mathcal{A}'_i = \{s_c\} \cap \{s_i\}$ of case **D**. As $s_k, s'_k \in L$, $s_k \neq s_1, s_{84}$, we have that $O(L) \cong 4$. If $O(L) = 4$, then all double homothetisms of U have as second components s_1, s_{84}, s_k or s'_k , which is again case c_1 , hence $O(L) \cong 4$. Then the endomorphisms $s_1, s_{84}, s_k, s'_k, s_l, s'_l$ occur in L , where $s_l \neq s_1, s_{84}, s_k, s'_k$ and $s_l \in \{s_c\} \cap \{s_i\}$. An element s_a that commutes with all elements of $\mathcal{A}'_i$ belongs to $\mathcal{B}'_i$, where $\mathcal{B}'_i = \{s_k\} \cap \{s_l\}$. Hence $U \subseteq S_i^\wedge$, where $S_i^\wedge$ is a maximal ring of related double homothetisms of R of case **D**. It follows that $U = S_i^\wedge$. The holomorph corresponding to U is one of the holomorphs U_i ($i = 1, \dots, 84$). In the preceding cases a, b and c we have assumed the existence of an element $(s_i, s_k) \in U$ with $s_i \neq s_1, s_{84}$ and $s_k \neq s_1, s_{84}$. Now every endomorphism $s_n \in E(R^+)(s_n \neq s_1, s_{84})$ belongs either to a set $\{s\}$ of case (i) of **A** or to at least one set $\{s'\}$ of case (ii) of **A**. If s_i and s_k belong to the *same* set $\{s\}$ of case (i) of **A**, we get case a . The case that s_i and s_k belong to *different* sets of case (i) of **A** is impossible, as $s_i s_k = s_k s_i$. It is a consequence of properties I and II that $s_i \in \{s\}$ of case (i) of **A** and $s_k \in \{s'\}$ of case (ii) of **A** or conversely is also impossible. There remains the case that s_i and s_k both belong to sets of case (ii) of **A**. If s_i and s_k belong to the *same* set $\{s'\}$ of case (ii) of **A**, we get cases b and c . Suppose now that $s_i \in \{s\}$ and $s_k \in \{s'\}$, where both $\{s\}$ and $\{s'\}$ are sets of case (ii) of **A**, such that $\{s\} \neq \{s'\}$. Then $O(\{s_i\}) = O(\{s_k\}) = 8$ is impossible, since this would imply $\{s\} = \{s'\}$ by property II. If at least one of the sets $\{s_i\}$ and $\{s_k\}$ has order 32, we can proceed as in the cases b_2 or c , since in b_2 and in c the assumption that s_i and s_k belong to the *same* set of case (ii) of **A** is not essential.

d . Suppose there is no element $(s_i, s_k) \in U$ with $s_i \neq s_1, s_{84}$ and $s_k \neq s_1, s_{84}$. It is clear that, if $(s_1, s_c) \in U$, $(s_{84}, s_c) \in U$ and, if $(s_c, s_1) \in U$, $(s_c, s_{84}) \in U$. Now suppose (s_1, s_t) and (s_u, s_1) belong to U , where $s_t \neq s_1, s_{84}$ and $s_u \neq s_1, s_{84}$. If (s_a, s_b) is an arbitrary element of U , then $s_a \in \{s_t\}$ and $s_b \in \{s_u\}$. If $\{s_t\}$ and $\{s_u\}$ both have order 8, then $\{s_t\} = \{s_u\}$ (property II) and this means that U is a *proper* subset of a maximal ring of **A** or **B**, which contradicts the maximality of U . Also if either $\{s_t\}$ or $\{s_u\}$ or both have order 32, we can prove that U is a *proper* subring of a maximal ring of related double homothetisms of R of one of the cases **A, B, C** or **D** by the same method as in case c with only a few modifications. As U is maximal this gives a contradiction at any case. Hence $(s_1, s_t), (s_u, s_1) \in U$ with $s_t \neq s_1, s_{84}$ and $s_u \neq s_1, s_{84}$ is impossible. Thus, if $(s_1, s_t) \in U$ with $s_t \neq s_1, s_{84}$, then for every element $(s_a, s_b) \in U$ one gets $s_a = s_1$ or $s_a = s_{84}$. But then $U \subseteq S^{iv}$, and as U is maximal, $U = S^{iv}$. If there is no element $(s_1, s_t) \in U$ with $s_t \neq s_1, s_{84}$, then for every element $(s_a, s_b) \in U$ one has $s_b = s_1$ or $s_b = s_{84}$. In that case $U \subseteq S^v$, and as U is maximal, $U = S^v$. The only holomorphs corresponding to rings U of d are V_1 and V_2 . This completes the proof that the zero-ring R with additive group of type $(2, 2, 2)$ has 367 holomorphs.

In [1] we have determined the number of holomorphs of all rings R with additive group R^+ of type (p, p) and (p, p^2) , where p is a prime number. Together with our previous results for rings R with R^+ of type $(2, 2, 2)$ we can give a survey of the number of holomorphs of all rings R with $O(R) < 16$. If R has 2, 3, 5, 7, 11 or 13 elements, then the additive group R^+ of R is a cyclic one, and as $E(R^+)$ is com-

mutative, R has one holomorph [1]. If R has 6, 10, 14 or 15 elements, then, as is well known, the additive group R^+ of R is again cyclic and R has one holomorph. In the sequel we leave the rings R with cyclic additive group R^+ out of consideration. If R has 4 resp. 9 elements, then the additive group R^+ of R is of type (p, p) with $p=2$ resp. 3. In this case, if R is a non-zero ring, R has one holomorph (Satz 2, [1]). A zero-ring R with R^+ of type $(2, 2)$ has $2^2 + 2 + 3 = 9$ holomorphs. A zero-ring R with R^+ of type $(3, 3)$ has $3^2 + 3 + 3 = 15$ holomorphs. If R has 12 elements and R^+ is not a cyclic group, then R^+ is the direct sum of a four-group of Klein and a group of order 3. Therefore R is the ring-theoretical direct sum of its subrings R_1 resp. R_2 , consisting resp. of the 4 elements of order 2 and the 3 elements of order 3 in R . As both R_1 and R_2 are characteristic subrings in R all holomorphs of R have the form $H = H_1 \oplus H_2$, where H_i is an arbitrary holomorph of R_i ($i=1, 2$) [2]. As R_2 has one holomorph it follows that R has one holomorph if R_1 has one holomorph. If R_1 is a non-zero ring it has one holomorph. If R_1 is a zero-ring it has 9 holomorphs as we have seen. Consequently, a ring R with 12 elements has one holomorph, except when the subring of elements of order 2 in R is a zero-ring. In the last case R has 9 holomorphs.

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