Further examples of normal numbers

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Introduction

Definition 1. 1. A number α is simply normal to the base (or scale) r iff, in the expansion of the fractional part of α to the base r, we have

$$\lim_{n\to\infty} \frac{n_c}{n} = \frac{1}{r},$$

for all c, where n_c is the number of occurrences of the digit c in the first n digits of α.

Definition 1. 2. A number is normal to the base r iff $\alpha, r\alpha, r^2\alpha, \dots$ are each simply normal to all the bases $r, r^2, r^3, \dots$.

Definition 1. 3. A real irrational number α is said to be a Liouville number iff, for every positive integer n, there is a rational number p/q, with $q>1$, depending on n, such that

$$|\alpha - p/q| < 1/q^n.$$

We shall call the class of such α, L.

It was first proved by E. BOREL [1], who introduced the concept of normal numbers, as above, that almost all real numbers are normal.

There are, however, only a very few numbers that have been proved normal. The first of these is known as Champernowne's number $x = .1234567891011121314\dots$, formed by writing the natural numbers in succession. CHAMPERNOWNE [2] showed this number normal to the base ten, while COPELAND and ERDŐS [3] extended the proof for normality to any base r of the analogous number constructed as above. TH. SCHEIDER [7] shows that Champernowne's number is transcendental but not in L.

We will introduce the class $\mathscr{L}$, which yields normality when its elements are added to normal numbers. This generates a set of normal numbers having the power of the continuum, and different from the known normal numbers. The remainder of the paper will partially characterize the class $\mathscr{L}$.

The class $\mathscr{L}$

Definition 2. 1. Let c be a given digit (or finite sequence of length s of digits) to the scale r, and let λ be a number to the scale r such that the number of non-c digits in the first N digits (sN digits) of λ is $f(N) = o(N)$.

Definition 2. 2. We define $\mathscr{L}$ to be the class of all elements of the type λ.
Note: The class $\mathscr{L}$ is closed under addition and subtraction.

Theorem 2.1 If λ is an element of $\mathscr{L}$, and if α is normal to the scale r, then $\alpha+(p/q)\lambda+p_1/q_1$ is normal to the scale r, where p, q, p_1, q_1 are integers.

PROOF. It is necessary to consider only the case when $\gamma = \alpha+\lambda$, since $\beta=(q/p)\alpha$ is normal. Thus, proving the theorem for $\beta+\lambda$ gives us the fact that

$$(p/q)(\beta+\lambda) = (p,q)[(q/p)\alpha+\lambda] = \alpha+(p/q)\lambda$$

is normal. Further, we need only consider λ of the form when c is zero, since $\lambda-.ccc...$ is in $\mathscr{L}$, with the dominant digit zero.

Let A_s be a given block of s digits to the scale r. We wish to classify the occurrences of A_s in α according to how many digits equal to $r-1$ follow A_s before a non-$r-1$ digit occurs. Hence, define A_{s+j} to be a block of $s+j+1$ digits, equal to $r-1$, followed by a non-$r-1$ digit. Since, for each k, the fractional part of r^k is less than one, the part to be carried in adding α and λ will never exceed one. Thus, a non-zero digit from λ, added into α to the right of A_{s+j} does not change the A_s part. Further, any digit of λ, added *into* the block A_{s+j} can change no more than the one A_s block occurring in that A_{s+j}.

Now, we fix $j\geqq 0$ and A_s, and from the normality of α and a result of NIVEN-ZUCKERMAN [6], we find that

$$\lim_{N\to\infty}\frac{N(A_{s+j})}{N}=\frac{1}{r^{s+j+1}},$$

for each A_{s+j}, where $N(A_{s+j})$ stands for the number of occurrences of the blocks A_{s+j} in the first N digits of α. But there are $r-1$ possible assignments to the non-zero digit terminating A_{s+j}; hence

$$\lim_{N\to\infty}\frac{N(A_{s+j})}{N}=\frac{r-1}{r^{s+j+1}},$$

for all A_{s+j}.

Thus for any $\varepsilon>0$ and $t>0$, we may find an M, such that

$$\left|\frac{N(A_{s+j})}{N}-\frac{r-1}{r^{s+j+1}}\right|<\frac{\varepsilon}{2t},$$

if $N>M$; or, in particular, we have

$$-\frac{N\varepsilon}{2t}<N(A_{s+j})-\frac{r-1}{r^{s+j+1}}N.$$

If $N'(A_{s+j})$ represents the number of these A_s blocks within A_{s+j} blocks that are preserved in the first N digits of $\gamma=\alpha+\lambda$, we have

$$N'(A_{s+j})+f(N)\geqq N(A_{s+j}),$$

and

$$-\frac{N\varepsilon}{2t}-f(N)<N'(A_{s+j})-\frac{r-1}{r^{s+j+1}}N.$$

Thus

$$\sum_{j=0}^{t}\left[-\frac{N\varepsilon}{2t}-f(N)\right] < \sum_{j=0}^{t}\left[N'(A_{s+j})-\frac{N(r-1)}{r^{s+j+1}}\right],$$

or

$$-\frac{N\varepsilon}{2}-tf(N) < \sum_{j=0}^{t} N'(A_{s+j})-\frac{N(r-1)}{r^{s+1}}\sum_{j=0}^{t}\frac{1}{r^j} \leqq \sum_{j=0}^{t} N'(A_{s+j})-\frac{N(r^{t+1}-1)}{r^{s+t+1}},$$

or finally,

$$\sum_{j=0}^{t}\frac{N'(A_{s+j})}{N}-\frac{r^{t+1}-1}{r^{s+t+1}} > -\frac{\varepsilon}{2}-\frac{tf(N)}{N}.$$

Since $\lim_{N\to\infty} f(N)/N=0$, we may choose N so large that $f(N)/N<\varepsilon/2t$, or $-\varepsilon/2t< -f(N)/N$. We now define $N'(A_s)$ to be the number of occurrences of A_s blocks in the first N digits of γ. Then, since $N'(A_s)\geqq \sum_{j=0}^{t} N'(A_{s+j})$, we have for some N,

$$\sum_{s=0}^{t} r' \frac{(A_{s+j})}{N}-\frac{1}{r^s} > -\frac{\varepsilon}{2}-t\left(\frac{\varepsilon}{2t}\right)-\frac{1}{r^{s+t+1}},$$

or

$$\frac{N'(A_s)}{N}-\frac{1}{r^s} > -\varepsilon-\frac{1}{r^{s+t+1}}.$$

Thus,

$$\liminf_{N\to\infty}\left(\frac{N'(A_s)}{N}-\frac{1}{r^s}\right) \geqq -\frac{1}{r^{s+t+1}},$$

and the left member is independent of t, while the right is as small as one pleases, which implies that

$$\liminf_{N\to\infty}\left(\frac{N'(A_s)}{N}-\frac{1}{r^s}\right) \geqq 0.$$

From lemma 8. 2 of [5], it follows that

$$\lim_{N\to\infty}\left(\frac{N'(A_s)}{N}-\frac{1}{r^s}\right) = 0,$$

and using the Niven-Zuckerman result once more, it follows that $\gamma = \alpha+\lambda$ is normal.

A partial characterization of the class $\mathscr{L}$

We state two lemmas which are not difficult to prove.

Lemma 3.1. *If p/q is a rational number such that $(p, q)=1$, and if there exists a rational number p_1/q_1, $(p_1, q_1)=1$, satisfying the relationship*

$$\frac{p}{q} < \frac{p_1}{q_1} < \frac{p}{q}+\frac{1}{mq},$$

where m is an integer $\geqq q-1$, then $q_1\geqq q$. Further, the condition $m\geqq q-1$ is necessary.

Lemma 3.2. *If p/q is a rational number, $(p, q)=1$, satisfying*

$$\frac{t\,10^{s+1}+1}{10^{2s+1}} < \frac{p}{q} < \frac{t\,10^{s+1}+1}{10^{2s+1}} + \frac{1}{q^n},$$

for integers n, s, and t such that $(t, 10^{2s+1})=1$, and $q^n > 10^{2s-1}$, then $q \geqq 10^{s-2}$.

PROOF. There is some rational fraction p_1/q_1 such that

$$\frac{p}{q} = \frac{t\,10^{s+1}+1}{10^{2s+1}} + \frac{p_1}{q_1},$$

which gives us $p_1/q_1 < 1/q^n < 1/10^{2s-1}$, or $p_1 10^{2s-1} < q_1$. Further combining the right hand side of this equality, we get

$$\frac{p}{q} = \frac{q_1(t\,10^{s+1}+1)+p_1\,10^{2s+1}}{q_1\,10^{2s+1}}.$$

Now if q_1 contains a power of ten, not equal to $2s+1$, we are done; for if $q_1 = 10^v q'$, where $10 \nmid q'$, and $v = 2s+1+r$ for $r > 0$, then

$$\frac{p}{q} = \frac{10^{2s+1+r}q'(t\,10^{s+1}+1)+p_1\,10^{2s+1}}{10^{2s+1}\,10^{2s+1+r}q'}$$

$$= \frac{10^r q'(t\,10^{s+1}+1)+p_1}{10^{2s+1+r}q'}.$$

But at most, we can divide 10^r out of both numerator and denominator of this last fraction, when $p_1 = 10^{r+u}p_2$ for some positive integer p_2, and $u \geqq 0$; and at least, this fraction is in lowest terms when $(p_1, 10)=1$, since $(p_1, q_1)=1$, and hence $(p_1, q')=1$. Then we have $q \geqq 10^{2s+1}q'$, and our result follows immediately.

Similarly, if $q_1 = 10^v q'$, where $10 \nmid q'$, and $v = 2s+1-r$ for some positive $r \leqq 2s+1$, then

$$\frac{p}{q} = \frac{q'\,10^{2s+1-r}(t\,10^{s+1}+1)+p_1\,10^{2s+1}}{10^{2s+1}\,10^{2s+1-r}q'}$$

$$= \frac{q'(t\,10^{s+1}+1)+p_1\,10^r}{2^{s+1}q'}.$$

And the most that can happen to this fraction is its reduction by some power of two or some power of five, when and only when q' is a multiple of this common divisor. In any case, our power of ten in the denominator remains intact, and $q \geqq 10^{2s+1}$, yielding the desired result.

We suppose, finally, that $v = 2s+1$. Then

$$\frac{p}{q} = \frac{10^{2s+1}q'(t10^{s+1}+1)+p_1 10^{2s+1}}{10^{2s+1}10^{2s+1}q'} =$$

$$= \frac{q'(t10^{s+1}+1)+p_1}{10^{2s+1}q'} =$$

$$= \frac{tq'10^{s+1}+(q'+p_1)}{10^{2s+1}q'}.$$

Now assume $q' < 10^{s-2}$. We have $p_1 10^{2s-1} < q_1$, and $q_1 = 10^{2s+1}q'$, so $p_1 < 10^2 q'$. Then

$$q'+p_1 < q'+10^2 q' = 101q' < 10^3 q' < 10^3 10^{s-2} = 10^{s+1}.$$

Thus, with an argument analogous to those preceding, we find that not quite as much as 10^{s+1} could possibly be factored out of both numerator and denominator of our fraction, and hence,

$$q > \frac{10^{2s+1}q'}{10^{s+1}} = 10^s q',$$

and our result follows.

Assuming $q' \geqq 10^{s-2}$, we find by an argument similar to that of the preceding lemma that the maximum divisor of both numerator and denominator is 10^{2s+1}, giving us $q \geqq q' \geqq 10^{s-2}$.

We are now ready to look closely at the class $\mathscr{L}$. First we make the following definition of E. MAILLETT [4]:

Definition 3. 1. A real number

$$x = A + \sum_{n=1}^{\infty} \frac{\delta_n}{q^{\psi(n)}},$$

(where δ_n is a positive integer $\leqq q-1$, A and q are positive integers, and $\psi(n)$ is a monotone increasing function of n, taking on integral values), is a quasi-rational number iff x, when represented to the base q, contains after the $\psi(n)^{\text{th}}$ digit, δ_n, followed by an increasing number of zeros. We will show, in the remainder of this paper:

Theorem 3.1. $\mathscr{L} \cap L \neq \emptyset$.

Theorem 3.2. $\mathscr{L} \not\subset L$.

Theorems 3.3. and 3.4. $L \not\subset \mathscr{L}$.

Note that $R \subset \mathscr{L}$, where R represents the rationals, and $R \supset$ quasirationals.

PROOF of *Theorem* 3. 1. Take

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{10^{n!}} = .110001000000000000000000010\ldots$$

It is clear that α is in $\mathscr{L}$, since $\lim_{N\to\infty} f(N)/N=0$, where $f(N)$ is the number of occurrences of the digit "1" in the first N digits of α. It is also well known that α is Liouville.

PROOF of *Theorem* 3. 2. Consider $\alpha=.1010001000000000100\ldots=\sum_{n=1}^{\infty} 10^{-\sum_{i=1}^{n-1} 2^i}$

Again, α is in class $\mathscr{L}$. The proof that α is not in L is by contradiction. Thus, we assume α is Liouville; hence that, for every positive integer n, there is a rational p/q, dependent upon n, $q>1$, such that $|\alpha-p/q|<1/q^n$. This equality may be further written as $p/q-1/q^n<\alpha<p/q+1/q^n$, and we will consider two cases: first if $p/q<\alpha<p/q+1/q^n$ for some $n\geqq 3$, and second, when $p/q-1/q^n<\alpha<p/q$.

In both of these cases, we will use the following notation:

We wish to keep track of the digits in equalling "1". Thus

(i) the k^{th} occurrence of a one will be in the q_k^{th} digit $=10^{e_k}$, where $e_k=\sum_{i=0}^{k-1} 2^i$.

Notice that

$$e_k=\sum_{i=0}^{k-2} 2^i+2^{k-1}=$$

$$=2\sum_{i=1}^{k-2} 2^{i-1}+2^{k-1}+1=$$

$$=2\sum_{i=0}^{k-3} 2^i+2\cdot 2^{k-2}+1=$$

$$=2\sum_{i=0}^{k-2} 2^i+1=2e_{k-1}+1.$$

(ii) α_k is the partial representation of α up to and including the k^{th} occurrence of a one, followed only by zeros.

(iii) $\alpha_k=p_k/q_k$.

Using this notation, we see that $\alpha_1<\alpha_2<\ldots<\alpha$ for all k and $\lim_{k\to\infty}\alpha_k=\alpha$.

Case 1. Here, $p/q<\alpha<p/q+1/q^n$. We may restrict our consideration to $n\geqq 3$.

Since p/q is less than α, and p/q is rational, we find that p/q and α agree in representation, digit by digit, until the q_k^{th} digit, where α contains a "1", but p/q contains a "0", and is followed by any sequence of digits. Thus, using the notation above, we have

$$\alpha_{k-1}\leqq p/q<\alpha_k, \quad \text{for some integer } k. \tag{1}$$

But, if $\alpha_{k-1}=p_{k-1}/q_{k-1}=p/q$, since both are reduced to their lowest terms, it follows that $q_{k-1}=q=10^{e_{k-1}}$, and $p_{k-1}=p$, and recalling that $e_k=2e_{k-1}+1$, the only way for $p/q+1/q^n$ to exceed α is for $n\leqq 2$.

(This is true, since p_{k-1}/q_{k-1} agrees with α for the first q_{k-1} digits, and one can multiply by *at most* $10^{e_{k-1}}$ and add a one in order to exceed α. That is, the least addition to p_{k-1}/q_{k-1} will be a string of zeros, than a one, the digital length of which equals that of p_{k-1}, which is equivalent to squaring the base.)

This equality is impossible under our assumption that $n \geqq 3$, and (1) becomes

$$\alpha_{k-1} < p/q < \alpha_k$$

or

(2) $$p_{k-1}/q_{k-1} < p/q < p_k/q_k.$$

Now $q_k = 10^{e_k} = 10^{2e_{k-1}+1} = 10^{e_{k-1}} \cdot 10^{e_{k-1}+1}$, and $p_k = p_{k-1} 10^{e_{k-1}+1} + 1$, which implies

$$\frac{p_k}{q_k} = \frac{p_{k-1} 10^{e_{k-1}+1} + 1}{10^{e_{k-1}} 10^{e_{k-1}+1}}$$

$$= \frac{p_{k-1}}{q_{k-1}} + \frac{1}{q_{k-1} 10^{e_{k-1}+1}}.$$

Letting $m = 10^{e_{k-1}+1}$ and using the above equality, we may write (2) as

$$\frac{p_{k-1}}{q_{k-1}} < \frac{p}{q} < \frac{p_{k-1}}{q_{k-1}} + \frac{1}{q_{k-1}}.$$

It is evident that $m > q$, so we apply lemma 3. 1 to the above inequality to obtain $q \geqq q_{k-1}$. Then

(3) $$1/q^n \leqq 1/q_{k-1}^n.$$

Further, we have $n \geqq 3$, so (3) becomes, after expansion:

$$1/q^n \leqq 1/q_{k-1}^n \leqq 1/q_{k-1}^3 = 1/10^{3e_{k-1}}.$$

And this implies that $-1/q^n$ affects at most, the $10^{3e_{k-1}-1\,st}$ digit in the expansion of α. That is, adding $1/q^n$ to p/q cannot affect the digits before the one containing the k^{th} "1", since this occurs in the $10^{2e_{k-1}-1\,st}$ digit. But, the only way p/q can be made to exceed α is to change one of the zeros in the first q_k digits of p/q to a one.

Therefore, $1/q^n + p/q < \alpha$, which contradicts the assumption that α is Liouville, and $\alpha < p/q + 1/q^n$. Thus α is not Liouville in this case.

Case 2. We now consider $p/q - 1/q^n < \alpha < p/q$. But case 1 implies that for all $n \geqq 3$, if α lies in the right half of the interval $(p/q - 1/q^n, p/q + 1/q^n)$, we do not have α Liouville; so if α is Liouville, α must lie in the left half of the above interval for all $n \geqq 3$. It is then necessary to find only one n implying a contradiction to deny our assumption that α is Liouville.

Using the notation as before, we find there is some integer k, such that

$$\alpha_{k-1} \leqq p/q - 1/q^n < \alpha_k.$$

Now, if $\alpha_{k-1} = p/q - 1/q^n$, then $q_{k-1} = q^n = 10^{e_{k-1}}$, since $(pq^{n-1} - 1, q^n) = 1$. But, adding $1/q^n = 1/10^{e_{k-1}}$ to α_{k-1} changes α_{k-1} only in the last non-zero digit, which becomes a two, and so the sum, equal to p/q, is not in lowest terms, since $(p, q) = 2$. (One cannot reduce p and q further, for there is no way for $(q/2)^n$ to equal $10^{e_{k-1}}$.)

Hence, we have the inequality

$$\alpha_{k-1} < p/q - 1/q^n < \alpha_k$$

or

(4) $$\alpha_{k-1} < \alpha_{k-1} + 1/q^n < p/q < \alpha_k + 1/q^n.$$

By the construction of α, the only way for p/q to exceed α is for $1/q^n$ to exceed $1/10^{e_k}$, or more appropriately, for the inequality

$$\frac{1}{q^n} > \frac{1}{10^{2e_k-1}}$$

to hold. Then we may apply lemma 3. 2 to (4), and let

$$q \geqq 10^{e_k-2} = 10^{2e_{k-1}-1},$$

and the conclusion follows exactly as that of case 1.

Thus we have shown α not Liouville.

To demonstrate a Liouville number that is not in class $\mathscr{L}$, we consider the number

$$\alpha = .110001\overbrace{1\ldots1}^{17}\overbrace{10\ldots0}^{120-24}\overbrace{11\ldots1}^{720-120}10\ldots$$
$$= \lim_{n\to\infty} p_n/q_n,$$

where

$$q_n = 10^{n!},$$
$$p_1 = 1,$$
$$p_{2k} = p_{2k-1}10^{(2k)!-(2k-1)!} + \sum_{i=0}^{(2k)!-(2k-1)!-1} 10^i$$
$$p_{2k+1} = p_{2k}10^{(2k)!-(2k-1)!} + 1.$$

We wish to show first that

Theorem 3. 3. *α is Liouville.*

PROOF. This is not difficult to show.

Theorem 3. 4. The α in theorem 3. 3 is not in class $\mathscr{L}$.

PROOF. Since α is the limit of a sequence of rationals, we may write α independently as the limit of two subsequences,

$$\alpha = \alpha_1 = \lim_{k\to\infty} \frac{p_{2k-1}}{q_{2k-1}},$$

and

$$\alpha = \alpha_2 = \lim_{k\to\infty} \frac{p_{2k}}{q_{2k}}.$$

If we count the number of occurrences of the non-zero digit one in the first q_{2k} digits of α_2, we find this frequency has a superior limit $=1$. But, counting this frequency in the first q_{2k-1} digits of α_1 gives us an inferior limit $= 0$. Thus, in α,

$$\lim_{N\to\infty} \sup \frac{f(N)}{N} \neq \lim_{N\to\infty} \inf \frac{f(N)}{N},$$

where $f(N)$ is the number of non-zero digits occurring in the first N digits of α; and hence,

$$\lim_{N\to\infty} \frac{f(N)}{N} \quad \text{does not exist.}$$

We have shown that α is not in class $\mathscr{L}$.

In passing, we remark that the number α, just defined, is also a quasi-rational number.

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