

On generalized convolution quotients

To Professor B. Gyires on his 60th birthday

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Intrroduction

Let $\mu(x)$ be a continuous increasing function in $\alpha < x < \beta$ ($-\infty \leq \alpha < \beta \leq \infty$) such that $\lim_{x \rightarrow \alpha+0} \mu(x) = -\infty$ and $\lim_{x \rightarrow \beta-0} \mu(x) = \infty$. $\mu(x)$ will be referred to as a base function on (α, β) . Let $\mathcal{C}^+(\alpha, \beta)$ be the linear space of continuous functions on (α, β) which vanish in some right-sided neighbourhood of α . By means of $\mu(x)$ we define the generalized convolution in $\mathcal{C}^+(\alpha, \beta)$ as follows

$$(1) \quad f * g = \int_{\alpha}^{\beta} f[\mu^{-1}(\mu(x) - \mu(t))] g(t) d\mu(t) \quad (f, g \in \mathcal{C}^+(\alpha, \beta)).$$

In the case of $\mu(x) = x$, (1) becomes the usual convolution

$$\int_{-\infty}^{\infty} f(x-t) g(t) dt$$

in $\mathcal{C}^+(-\infty, \infty)$. If $\mu(x) = \log x$, then (1) is limited to the multiplicative convolution

$$\int_0^{\infty} f\left(\frac{x}{t}\right) g(t) \frac{1}{t} dt.$$

We prove that the space $\mathcal{C}^+(\alpha, \beta)$ endowed with the multiplication (1) is a ring without divisors of zero. This ring will be denoted by $\mathcal{C}_{\mu}(\alpha, \beta)$. Every $\mathcal{C}_{\mu}(\alpha, \beta)$ is isomorphic to $\mathcal{C}_{\{x\}}(-\infty, \infty)$. The quotient field of $\mathcal{C}_{\mu}(\alpha, \beta)$ will be denoted by $\mathcal{M}_{\mu}$, and it will be called the field of generalized convolution quotients. $\mathcal{M}_{\{x\}}$ is the field of Mikusiński operators. Every $\mathcal{M}_{\mu}$ is isomorphic to $\mathcal{M}_{\{x\}}$. § 1 is devoted to the definition and basic properties of the field $\mathcal{M}_{\mu}$. In § 2 the embedding of the locally Stieltjes integrable functions in $\mathcal{M}_{\mu}$ is given. In § 3 the properties of the shift operator in $\mathcal{M}_{\mu}$ is investigated. In § 4 the convergence is defined. Finally, in § 5, linear transformations of $\mathcal{M}_{\mu}$ are considered. Thus, it will be showed that every $\mathcal{M}_{\mu}$ gives an operational calculus for the operator $d/d\mu(t)$.

§ 1. The field $\mathcal{M}_\mu$

Definition 1.1. A real valued function $\mu(x)$, defined in $\alpha < x < \beta$ ($-\infty \leq \alpha < \beta \leq \infty$), is called a base function if

- 1° $\mu(x)$ is continuous in $\alpha < x < \beta$,
- 2° $\mu(x)$ is increasing in $\alpha < x < \beta$,
- 3° $\lim_{x \rightarrow \alpha+0} \mu(x) = -\infty$,
- 4° $\lim_{x \rightarrow \beta-0} \mu(x) = \infty$.

We shall show that every base function $\mu(x)$ determines a field $\mathcal{M}_\mu$, which is isomorphic to the field $\mathcal{M}$ of the Mikusiński operators.

Definition 1.2. Let $\alpha < \xi < \beta$ and let $\mathcal{C}^\xi(\alpha, \beta)$ be the class of the complex-valued functions, defined on $\alpha < x < \beta$ such that every function $f(x)$ of $\mathcal{C}^\xi(\alpha, \beta)$ is continuous in $\alpha < x < \beta$ and vanishes in $\alpha < x \leq \xi$. Let

$$\mathcal{C}^+(\alpha, \beta) = \bigcup_{\alpha < \xi < \beta} \mathcal{C}^\xi(\alpha, \beta)$$

Obviously, $\mathcal{C}^+(\alpha, \beta)$ is a vector space (under addition and multiplication by scalars).

It is well known ([1]) that the set $\mathcal{C}^+(-\infty, \infty)$ forms a commutative ring with respect to addition and multiplication in the following sense

$$(1.1) \quad f(x) * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t) dt, \quad (f, g \in \mathcal{C}^+(-\infty, \infty)).$$

The absence of divisors of zero in the ring $\mathcal{C}^+(-\infty, \infty)$ makes it possible to extend this ring to a quotient field $\mathcal{M}$ and the elements of the field $\mathcal{M}$ are called Mikusiński operators.

Definition 1.3. Let $\mu(x)$ be a base function in (α, β) and let $\mu^{-1}(t)$ be the inverse function of $\mu(x)$. The product of the functions $f(x) \in \mathcal{C}^+(\alpha, \beta)$ and $g(x) \in \mathcal{C}^+(\alpha, \beta)$ let be defined as follows

$$(1.2) \quad f(x) * g(x) = \int_{\alpha}^{\beta} f[\mu^{-1}(\mu(x) - \mu(t))]g(t) d\mu(t) \quad (x \in (\alpha, \beta)),$$

where the integral is understood in the sense of Stieltjes. The product (1.2) is called the generalized convolution of f and g .

Evidently, $f, g \in \mathcal{C}^+(\alpha, \beta)$ implies that $f * g \in \mathcal{C}^+(\alpha, \beta)$. The set $\mathcal{C}^+(\alpha, \beta)$ endowed with the operations of addition and multiplication defined by (1.2), forms an algebraic system and it will be denoted by $\mathcal{C}_\mu(\alpha, \beta)$.

The symbol $\{f(x)\}_\mu$ denotes that the function $f(x) \in \mathcal{C}^+(\alpha, \beta)$ is regarded as an element of the algebraic system $\mathcal{C}_\mu(\alpha, \beta)$. Thus we may preserve the usual notation of the algebraic operations in $\mathcal{C}_\mu(\alpha, \beta)$ without misunderstandings:

$$(1.3) \quad \{f(x)\}_\mu + \{g(x)\}_\mu = \{f(x) + g(x)\}_\mu$$

$$(1.4) \quad \{f(x)\}_\mu \{g(x)\}_\mu = \left\{ \int_{\alpha}^{\beta} f[\mu^{-1}(\mu(x) - \mu(t))]g(t) d\mu(t) \right\}_\mu$$

Remark 1.1. The function $\mu(x)=x$ is a base function in $(-\infty, \infty)$. In this case the product (1.2) reduces to the convolution (1.1). The term "generalized convolution" is motivated by the above circumstance. In the case of $\mu(x)=x$ we shall preserve the original notations of Mikusiński and we shall write simply

$$(1.5) \quad \{f(t)\}\{g(t)\} = \left\{ \int_{-\infty}^{\infty} f(t-\tau)g(\tau) d\tau \right\}$$

and $\mathcal{C}^+(-\infty, \infty)$ will denote the convolution ring in which the multiplication is defined by (1.5).

Theorem 1.1. *Let $\mu(x)$ be a base function in (α, β) . Then the set $\mathcal{C}_\mu(\alpha, \beta)$ forms a commutative ring without zero divisors. Moreover, $\mathcal{C}_\mu(\alpha, \beta)$ is isomorphic to the convolution ring $\mathcal{C}^+(-\infty, \infty)$ (see remark 1.1).*

PROOF. It will be sufficient to prove that the mapping

$$(1.6) \quad L_\mu\{f(t)\} = \{f[\mu(x)]\}_\mu$$

of $\mathcal{C}^+(-\infty, \infty)$ onto $\mathcal{C}_\mu(\alpha, \beta)$ is an isomorphism. It is clear that (1.6) defines a one-to-one correspondence between the elements of $\mathcal{C}^+(-\infty, \infty)$ and $\mathcal{C}_\mu(\alpha, \beta)$. We shall show that the mapping (1.6) preserves the algebraic operations. In fact:

$$\begin{aligned} L_\mu(\{f(t)\} + \{g(t)\}) &= L_\mu\{f(t) + g(t)\} = \{f[\mu(x)] + g[\mu(x)]\}_\mu = \\ &= \{f[\mu(x)]\}_\mu + \{g[\mu(x)]\}_\mu = L_\mu\{f(t)\} + L_\mu\{g(t)\}, \end{aligned}$$

and

$$\begin{aligned} L_\mu(\{f(t)\}\{g(t)\}) &= L_\mu\left\{ \int_{-\infty}^{\infty} f(t-\tau)g(\tau) d\tau \right\} = \left\{ \int_{-\infty}^{\infty} f[\mu(x)-\tau]g(\tau) d\tau \right\}_\mu = \\ &= \left\{ \int_{\alpha}^{\beta} f[\mu(x)-\mu(y)]g[\mu(y)] d\mu(y) \right\}_\mu = \left\{ \int_{\alpha}^{\beta} f[\mu(\mu^{-1}(\mu(x)-\mu(y)))]g[\mu(y)] d\mu(y) \right\}_\mu = \\ &= \{f[\mu(x)]\}_\mu \{g[\mu(x)]\}_\mu = L_\mu\{f(t)\} \cdot L_\mu\{g(t)\} \end{aligned}$$

Since $\mathcal{C}^+(-\infty, \infty)$ is a commutative ring which has no zero divisor, it follows from the isomorphism, that $\mathcal{C}_\mu(\alpha, \beta)$ is a commutative ring without zero divisors too. Thus the theorem is proved.

Corollary 1. The ring $\mathcal{C}_\mu(\alpha, \beta)$ can be extended in the usual way to a quotient field $\mathcal{M}_\mu$. The elements of $\mathcal{M}_\mu$ will be called generalized convolution quotients and denoted by

$$\frac{\{f(x)\}_\mu}{\{g(x)\}_\mu}, \dots \text{ etc.}$$

Corollary 2. Every element $\bar{a}$ of $\mathcal{M}_\mu$ has a representative

$$\bar{a} = \frac{\{f_0(x)\}_\mu}{\{g_0(x)\}_\mu}$$

where $\{f_0(x)\}_\mu$ and $\{g_0(x)\}_\mu$ are functions of the class $\mathcal{C}^{x_0}(\alpha, \beta)$ and $x_0 \in (\alpha, \beta)$ is the zero of $\mu(x) : \mu(x_0) = 0$.

Corollary 3. The mapping

$$\{f(x)\}_\mu \leftrightarrow \frac{\{f(x)\}_\mu \{g(x)\}_\mu}{\{g(x)\}_\mu} \quad (g \in \mathcal{C}_\mu(\alpha, \beta))$$

defines an embedding of $\mathcal{C}_\mu(\alpha, \beta)$ in $\mathcal{M}_\mu$.

Corollary 4. Let $\mathcal{K}$ be the field of complex numbers. The mapping

$$(1.7) \quad \lambda \leftrightarrow \frac{\{\lambda f(x)\}_\mu}{\{f(x)\}_\mu} \quad (\lambda \in \mathcal{K}, f \in \mathcal{C}_\mu(\alpha, \beta))$$

defines an embedding of $\mathcal{K}$ in $\mathcal{M}_\mu$.

Corollary 5. The field $\mathcal{M}_\mu$ is isomorphic to the field $\mathcal{M}$ of Mikusiński operators. This isomorphism is the extension of the mapping (1. 6) in the following manner:

$$(1.8) \quad L_\mu \left(\frac{\{f(t)\}}{\{g(t)\}} \right) = \frac{\{f[\mu(x)]\}_\mu}{\{g[\mu(x)]\}_\mu} = \frac{L_\mu(f)}{L_\mu(g)} \quad \left(\frac{f}{g} \in \mathcal{M} \right)$$

Remark 1. 2. We may regard the numbers on the one hand as elements of $\mathcal{M}$, on the other as elements of $\mathcal{M}_\mu$. We show that L_μ preserves the numbers:

$$(1.9) \quad L_\mu(\lambda) = \lambda \quad (\lambda \in \mathcal{K})$$

Indeed, let $f(t)$ be a function of $\mathcal{C}^+(-\infty, \infty)$, then $\tilde{f}(x) = f[\mu(x)]$ is a function of $\mathcal{C}_\mu(\alpha, \beta)$ and we have

$$L_\mu(\lambda) = L_\mu \left(\frac{\{\lambda f(t)\}}{\{f(t)\}} \right) = \frac{L_\mu\{\lambda f(t)\}}{L_\mu\{f(t)\}} = \frac{\{\lambda f[\mu(x)]\}_\mu}{\{f[\mu(x)]\}_\mu} = \frac{\{\lambda \tilde{f}(x)\}_\mu}{\{\tilde{f}(x)\}_\mu} = \lambda$$

§ 2. The embedding of locally integrable functions

Let $\mathcal{L}_\mu(\alpha, \beta)$ be the class of functions $f(x)$ defined in $\alpha < x < \beta$ such that

- (i) $f(x)$ vanishes identically in a right-sided neighbourhood of α .
- (ii) $f(x)$ is integrable with respect to $\mu(x)$ in every subinterval (α_1, β_1) ($\alpha \equiv \alpha_1 < \beta_1 < \beta$) in the sense of Lebesgue—Stieltjes.

If $\mu(x) = x$, then $\mathcal{L}_\mu(-\infty, \infty)$ is the class of the locally Lebesgue-integrable functions f vanishing in any interval $(-\infty, \lambda)$, where the number λ depends on f . We write in this case $\mathcal{L}^+(-\infty, \infty)$ instead of $\mathcal{L}_\mu(-\infty, \infty)$.

Theorem 2. 1. $f(x)$ is a function of $\mathcal{L}_\mu(\alpha, \beta)$ if and only if $f[\mu^{-1}(t)]$ is a function of $\mathcal{L}^+(-\infty, \infty)$.

This theorem is an immediate consequence of the well known connection between the Lebesgue and Lebesgue—Stieltjes integrals (see [2]).

By the embedding of $\mathcal{L}_\mu(\alpha, \beta)$ in $\mathcal{M}_\mu$ we make use of the fact that $\mathcal{L}^+(-\infty, \infty)$ is embedded in $\mathcal{M}$. Let $f(x) \in \mathcal{L}_\mu(\alpha, \beta)$. Then, by theorem 2. 1, $\{f[\mu^{-1}(t)]\} \in \mathcal{M}$. We identify the element $L_\mu\{f[\mu^{-1}(t)]\}$ of $\mathcal{M}_\mu$ with the function $f(x)$ and we write in this case

$$(2.1) \quad \{f(x)\}_\mu = L_\mu\{f[\mu^{-1}(t)]\}.$$

For $-\infty < \lambda < \infty$ let

$$(2.2) \quad H_\lambda(x) = \begin{cases} 0 & \text{if } x < \lambda \\ 1 & \text{if } \lambda \leq x. \end{cases}$$

Obviously, $H_\lambda(x)$ is a function of $\mathcal{L}_\mu(\alpha, \beta)$ and thus $\{H_\lambda(x)\}_\mu$ is an element of $\mathcal{M}_\mu$, provided $\alpha < \lambda < \beta$.

For the zero x_0 of $\mu(x)$, the function $l = \{H_{x_0}(x)\}_\mu$ is called the operator of integration with respect to $\mu(x)$. This definition is justified by

$$(2.3) \quad l\{f(x)\}_\mu = \left\{ \int_\alpha^x f(t) d\mu(t) \right\}_\mu$$

for $f \in \mathcal{L}_\mu(\alpha, \beta)$. Indeed, we obtain from (2.1) that

$$\begin{aligned} l\{f(x)\}_\mu &= L_\mu\{H_{x_0}[\mu^{-1}(t)]\} \cdot L_\mu\{f[\mu^{-1}(t)]\} = L_\mu(\{H_0(t)\} \{f[\mu^{-1}(t)]\}) = \\ &= L_\mu\left\{ \int_{-\infty}^{\infty} H_0(t-\tau) f[\mu^{-1}(\tau)] d\tau \right\} = L_\mu\left\{ \int_{-\infty}^t f[\mu^{-1}(\tau)] d\tau \right\} = \\ &= \left\{ \int_{\mu(\alpha)}^{\mu(x)} f[\mu^{-1}(\tau)] d\tau \right\}_\mu = \left\{ \int_\alpha^x f(t) d\mu(t) \right\}_\mu. \end{aligned}$$

Let $f(x)$ and $g(x)$ be defined in the neighborhood of the point x . The derivative of f with respect to g in the point x is the limit

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{g(x+h) - g(x)}$$

if it exists and it will be denoted by

$$(2.4) \quad \frac{df(x)}{dg(x)} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{g(x+h) - g(x)}.$$

If f and g have derivatives in x , then, obviously,

$$\frac{df(x)}{dg(x)} = \frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}} = \frac{f'(x)}{g'(x)}.$$

Theorem 2.2. Let $\tilde{f}(x)$ be a function of $\mathcal{C}_\mu(\alpha, \beta)$. If $\tilde{f}(x)$ is derivable with respect to $\mu(x)$ in all points of the interval $\alpha < x < \beta$, then the function

$$f(t) = \tilde{f}[\mu^{-1}(t)]$$

is also derivable in $-\infty < t < \infty$ and

$$(2.5) \quad \frac{d\tilde{f}(x)}{d\mu(x)} = f'[\mu(x)].$$

holds.

PROOF. Let $t = \mu(x)$. Since, by the continuity of $\mu(x)$, $\Delta t = \mu(x+h) - \mu(x) \rightarrow 0$ as $h \rightarrow 0$, we get

$$\begin{aligned} \frac{d\bar{f}(x)}{d\mu(x)} &= \lim_{h \rightarrow 0} \frac{\bar{f}(x+h) - \bar{f}(x)}{\mu(x+h) - \mu(x)} = \lim_{h \rightarrow 0} \frac{f[\mu(x+h)] - f[\mu(x)]}{\mu(x+h) - \mu(x)} = \\ &= \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = f'(t) = f'[\mu(x)], \end{aligned}$$

and the theorem is proved.

The element

$$(2.6) \quad \bar{s} = \frac{1}{l}$$

is called the operator of differentiation with respect to $\mu(x)$. This definition is motivated by the following theorem:

Theorem 2.3. Let $\bar{f}(x)$ be a function of $\mathcal{C}_\mu(\alpha, \beta)$. If $\frac{d\bar{f}}{d\mu}$ exists and $\frac{d\bar{f}}{d\mu} \in \mathcal{C}_\mu(\alpha, \beta)$, then

$$(2.7) \quad \bar{s}\{\bar{f}(x)\}_\mu = \left\{ \frac{d\bar{f}(x)}{d\mu(x)} \right\}_\mu.$$

PROOF. It follows from theorem 2.2 that $\{f'(t)\} \in \mathcal{C}^+(-\infty, \infty)$ for $f(t) = \bar{f}[\mu^{-1}(t)]$ and thus, by a known result of the Mikusiński's operational calculus (see [3] p. 192.),

$$s\{f(t)\} = \{f'(t)\} + f[\Lambda(f)]e^{-\Lambda(f)s}.$$

It follows from the continuity of f in $(-\infty, \infty)$ that $f[\Lambda(f)] = 0$. Therefore

$$(2.8) \quad s\{f(t)\} = \{f'(t)\}.$$

Since

$$\bar{s} = \frac{1}{l} = \frac{1}{\{H_{x_0}(x)\}_\mu} = \frac{1}{\{H_0[\mu(x)]\}_\mu} = \frac{L_\mu(1)}{L_\mu\{H_0(t)\}} = L_\mu\left(\frac{1}{\{H_0(t)\}}\right) = L_\mu(s),$$

we get

$$\bar{s}\{\bar{f}(x)\}_\mu = L_\mu(s)L_\mu(f) = L_\mu(sf) = L_\mu\{f'(t)\} = \{f'[\mu(x)]\}_\mu = \left\{ \frac{d\bar{f}(x)}{d\mu(x)} \right\}_\mu$$

and the theorem is proved.

§ 3. The shift operator

Definition 3.1. Let

$$\{H_\lambda(x)\}_\mu \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \alpha < x < \lambda \\ 1 & \text{if } \lambda \leq x < \beta, \end{cases} \quad (\alpha < \lambda < \beta).$$

The function

$$(3.1) \quad h(\lambda) = \bar{s}\{H_\lambda(x)\}_\mu$$

is called the *shift operator* in $\mathcal{M}_\mu$.

Theorem 3.1.

$$(3.2) \quad h(\lambda) = L_\mu(e^{-\mu(\lambda)s})$$

PROOF.

$$\begin{aligned} L_\mu(e^{-\mu(\lambda)s}) &= L_\mu(s\{H_{\mu(\lambda)}(t)\}) = L_\mu(s)L_\mu\{H_{\mu(\lambda)}(t)\} = \\ &= \bar{s}\{H_{\mu(\lambda)}[\mu(x)]\}_\mu = \bar{s}\{H_\lambda(x)\} = h(\lambda). \end{aligned}$$

Theorem 3.2. The function $h(\lambda)$ satisfies the functional equation

$$(3.3) \quad h(\mu^{-1}[\mu(\xi) + \mu(\eta)]) = h(\xi)h(\eta) \quad (\xi, \eta \in (\alpha, \beta)).$$

PROOF. It follows from theorem 3.1 that

$$\begin{aligned} h(\mu^{-1}[\mu(\xi) + \mu(\eta)]) &= L_\mu(e^{-[\mu(\xi) + \mu(\eta)]s}) = L_\mu(e^{-\mu(\xi)s}e^{-\mu(\eta)s}) = \\ &= L_\mu(e^{-\mu(\xi)s})L_\mu(e^{-\mu(\eta)s}) = h(\xi)h(\eta). \end{aligned}$$

Theorem 3.3. For $F \in \mathcal{C}_\mu(\alpha, \beta)$ and $\lambda \in (\alpha, \beta)$,

$$(3.4) \quad h(\lambda)\{F(x)\}_\mu = \{F(\mu^{-1}[\mu(x) - \mu(\lambda)])\}_\mu.$$

PROOF. If $f \in \mathcal{C}^+(-\infty, \infty)$, then $e^{-\lambda s}\{f(t)\} = \{f(t - \lambda)\}$. Thus

$$\begin{aligned} h(\lambda)\{F(x)\}_\mu &= L_\mu(e^{-\mu(\lambda)s})L_\mu(L_\mu^{-1}\{F(x)\}_\mu) = L_\mu(e^{-\mu(\lambda)s})L_\mu\{F[\mu^{-1}(t)]\} = \\ &= L_\mu(e^{-\mu(\lambda)s}\{F[\mu^{-1}(t)]\}) = L_\mu\{F[\mu^{-1}(t - \mu(\lambda))]\} = \{F(\mu^{-1}[\mu(x) - \mu(\lambda)])\}_\mu. \end{aligned}$$

§ 4. The convergence in $\mathcal{M}_\mu$

In this section we shall define the notion of the convergence in $\mathcal{M}_\mu$ and we shall show that the mapping L_μ is continuous.

Definition 4.1. A sequence of functions $f_n \in \mathcal{C}^+(\alpha, \beta)$ is said to be convergent in $\mathcal{C}^+(\alpha, \beta)$ to the function $f \in \mathcal{C}^\xi(\alpha, \beta)$ ($\xi \in (\alpha, \beta)$), if $f_n \in \mathcal{C}^\xi(\alpha, \beta)$ for all $n = 1, 2, \dots$ and if the sequence f_n is convergent to the limit f uniformly in any closed subinterval $[\xi, \eta]$ of $[\alpha, \beta]$. We write in this case

$$(4.1) \quad f_n \Rightarrow f \quad \text{in } \mathcal{C}^+(\alpha, \beta) \quad \text{as } n \rightarrow \infty.$$

Lemma 4.1. Let $\mu(x)$ be a base function in (α, β) . If $f_n \Rightarrow f$ in $\mathcal{C}^+(-\infty, \infty)$ as $n \rightarrow \infty$, then $L_\mu(f_n) \Rightarrow L_\mu(f)$ in $\mathcal{C}^+(\alpha, \beta)$ as $n \rightarrow \infty$.

PROOF. Let $f \in \mathcal{C}^\lambda(-\infty, \infty)$. Then $f_n \in \mathcal{C}^\lambda(-\infty, \infty)$ for all $n = 1, 2, \dots$. Consequently, $L_\mu(f_n) = \{f_n[\mu(x)]\}_\mu \in \mathcal{C}^\xi(\alpha, \beta)$ and $L_\mu(f) = \{f[\mu(x)]\}_\mu \in \mathcal{C}^\xi(\alpha, \beta)$ for $\xi = \mu^{-1}(\lambda)$. Fix $\varepsilon > 0$ and η such that $\xi < \eta < \beta$. Since the sequence of functions f_n is convergent to f uniformly in the segment $\lambda \leq t \leq \mu(\eta) < \infty$, therefore there is an integer N so that $|f_n(t) - f(t)| < \varepsilon$ whenever $n > N$ and $\lambda \leq t \leq \mu(\eta)$. Consequently,

$$|f_n[\mu(x)] - f[\mu(x)]| < \varepsilon$$

whenever $n > N$ and $\xi \leq x \leq \eta$. This proves the lemma.

Definition 4.2. A sequence of elements $A_n \in \mathcal{M}_\mu$ is said to be convergent in $\mathcal{M}_\mu$ to the limit $A \in \mathcal{M}_\mu$, if there exist representatives

$$\frac{F_n}{G_n} = A_n \quad (F_n, G_n \in \mathcal{C}_\mu(\alpha, \beta), \quad n = 1, 2, \dots)$$

and

$$\frac{F}{G} = A \quad (F, G \in \mathcal{C}_\mu(\alpha, \beta))$$

such that

$$F_n \Rightarrow F \quad \text{in } \mathcal{C}^+(\alpha, \beta) \quad \text{as } n \rightarrow \infty$$

and

$$G_n \Rightarrow G \quad \text{in } \mathcal{C}^+(\alpha, \beta) \quad \text{as } n \rightarrow \infty.$$

Theorem 4.1. If a sequence of operators a_n of the field $\mathcal{M}$ is convergent in $\mathcal{M}$ to the limit $a \in \mathcal{M}$, then the sequence of elements $L_\mu(a_n) \in \mathcal{M}_\mu$ is convergent in $\mathcal{M}_\mu$ to the limit $L_\mu(a)$:

$$(4.2) \quad L_\mu(a_n) \rightarrow L_\mu(a) \quad (n \rightarrow \infty).$$

PROOF. Let $\frac{f_n}{g_n} = a_n$ be a sequence of representatives such that $f_n \Rightarrow f$ and $g_n \Rightarrow g$ in $\mathcal{C}^+(-\infty, \infty)$ as $n \rightarrow \infty$. Then $\frac{f}{g} = a$ and, by lemma 4.1, we have $L_\mu(f_n) \Rightarrow L_\mu(f)$ and $L_\mu(g_n) \Rightarrow L_\mu(g)$ in $\mathcal{C}^+(\alpha, \beta)$ as $n \rightarrow \infty$. Thus, by definition 4.2,

$$L_\mu\left(\frac{f_n}{g_n}\right) = \frac{L_\mu(f_n)}{L_\mu(g_n)} \rightarrow \frac{L_\mu(f)}{L_\mu(g)} = L_\mu\left(\frac{f}{g}\right) \quad (n \rightarrow \infty)$$

and the theorem is proved.

Using theorem 4.1, the basic properties of the limit of a sequence in $\mathcal{M}_\mu$ can be easily deduced from the corresponding properties of the limit of a sequence in $\mathcal{M}$.

We remark that a similar theorem holds for L_μ^{-1} .

§ 5. Linear transformations of $\mathcal{M}_\mu$

We consider maps $\bar{\mathbf{F}}$ of $\mathcal{M}_\mu$ into $\mathcal{M}_\mu$. These are called transformations of $\mathcal{M}_\mu$.

Definition 5.1. Let $\mathbf{F}$ be an operator transformation of $\mathcal{M}$ (see [3]). The transformation

$$(5.1) \quad \bar{\mathbf{F}} = L_\mu \mathbf{F} L_\mu^{-1}$$

of $\mathcal{M}_\mu$ is called the equivalent of $\mathbf{F}$ in $\mathcal{M}_\mu$.

Theorem 5.1. If $\mathbf{F}$ and $\mathbf{G}$ are transformations of $\mathcal{M}$, then

$$\overline{\mathbf{F} + \mathbf{G}} = \bar{\mathbf{F}} + \bar{\mathbf{G}}$$

$$\overline{\mathbf{F}\mathbf{G}} = \bar{\mathbf{F}}\bar{\mathbf{G}}$$

$$\begin{aligned}\text{PROOF. } \overline{\mathbf{F} + \mathbf{G}} &= L_\mu(\mathbf{F} + \mathbf{G})L_\mu^{-1} = L_\mu(\mathbf{F}L_\mu^{-1} + \mathbf{G}L_\mu^{-1}) = \\ &= L_\mu\mathbf{F}L_\mu^{-1} + L_\mu\mathbf{G}L_\mu^{-1} = \overline{\mathbf{F}} + \overline{\mathbf{G}}\end{aligned}$$

and similarly

$$\overline{\mathbf{F}\mathbf{G}} = L_\mu(\mathbf{F}\mathbf{G})L_\mu^{-1} = (L_\mu\mathbf{F}L_\mu^{-1})(L_\mu\mathbf{G}L_\mu^{-1}) = \overline{\mathbf{F}}\overline{\mathbf{G}}.$$

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