New types of topological spaces

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§ 1. Introduction

Our aim, in the present paper, is to introduce some new types of topological spaces (see § 2). The definitions of these spaces are based on those of T_0, T_1 and T_2-spaces, (cf. [1][1]).

In § 4 below, we give some examples of these new spaces and in the rest of the paper we discuss some of their interesting properties and relations.

§ 2. The new definitions

Definition 2. 1. A topological space is a *T'_0-space* if and only if for each pair of distinct points of the space, there is an open neighborhood of one point to which the other is a boundary point[2]).

Definition 2. 2. A topological space is a *T'_1-space* if and only if for each pair x and y of distinct points of the space, there is an open neighborhood of one point, say x, to which y is a boundary point and a neighborhood of y to which x does not belong. (x is not necessarily a boundary point of the neighborhood of y.)

Definition 2. 3. A topological space is a *T''_1-space* if and only if for each pair x and y of distinct points of the space, there is an open neighborhood of x to which y is a boundary point.

Definition 2. 4.[3]) A topological space is a *T'_2-space* if and only if for each pair x and y of distinct points of the space, there exists disjoint open neighborhoods of x and y, whose boundary sets are disjoint.

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[1]) Numbers in square brackets refer to the bibliography at the end.

[2]) For the definition of the boundary of a subset of a topological space, see for example [1].

[3]) This definition could equivalently be stated as follows:

A topological space is a T'_2-space if and only if, for each pair x and y of distinct points of the space, there exists disjoint closed neighborhoods of x and y.

§ 3. Remarks

From the above definitions one may notice that:

(i) T_0'-space is a T_0-space,
(ii) T_1'-space is a $T_0' + T_1$-space,
(iii) T_1''-space is a T_1'-space,
(iv) A topological space with discrete topology satisfies none of the conditions T_0', T_1' and T_1'', because the boundary of every subset in a discrete topological space is void. Of course, it is a T_2'-space.
(v) A topological space with indiscrete topology is none of the given new spaces.
(vi) There is no T_1'-space or T_1''-space of finite elements. This because a T_1''-space or a T_1'-space is a T_1-space, but T_1-space of finite elements is discrete.
(vii) a T_2'-space is a T_2-space (a Hausdorff space).

So, one could draw the following diagram:

$$\begin{tikzcd}
T_2' \arrow[d, Rightarrow] & T_1'' \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & T_1' \arrow[r, Rightarrow] \arrow[dl, Rightarrow] & T_0' \arrow[d, Rightarrow, no head] \\
T_2 \arrow[rr, Rightarrow] & & T_1 \arrow[rr, Rightarrow] & & T_0
\end{tikzcd}$$

§ 4. Some examples of the new spaces[4])

We give now the following few examples of T_0', T_1', T_1'' and T_2'-spaces.

Example 1: The set of all real numbers with the usual topology is a T_0', T_1', T_1'', T_2''-space.

Example 2: Let X be a set consisting of four elements a, b, c and d. A topology can be defined in X by taking the open sets to be: (i) X itself, (ii) $\emptyset$ the void set, (iii) the subsets $\{c\}$, $\{b, c\}$, $\{a, b, c\}$ and $\{b, c, d\}$.
The topological space X so defined is a T_0'-space, but none of the other types.

Example 3: Let X be any infinite set. A topology can be defined in X by taking the open sets to be: (i) X itself, (ii) $\emptyset$; the void set, (iii) any set whose complement is finite. The topological space X is T_1' but not T_2'.

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[4]) The author has not yet found an example of a T_1'-space which is not a T_1''-space.

§ 5. Some properties of the new spaces

Theorem 5. 1. *A subset of a T_2'-spaces i, with the relative topology, a T_2'-space.*

PROOF. Let X be a T_2'-space and A be a subspace of X. Let x and y be any two distinct points of A. Then they are also points of X, and so, since X is T_2'-space, there are open sets U, V containing x and y respectively such that $\bar{U} \cap \bar{V} = \emptyset$. Let $W = U \cap A$ and $Z = V \cap A$. Then W and Z are open sets in A containing x and y respectively. We have also $\bar{W} \cap \bar{Z} = \emptyset$. Let[5]) $\tilde{W} = \bar{W} \cap A$ and $\tilde{Z} = \bar{Z} \cap A$. Then $\tilde{W} \cap \tilde{Z} = (\bar{W} \cap A) \cap (\bar{Z} \cap A) = (\bar{W} \cap \bar{Z}) \cap A = \emptyset \cap A = \emptyset$.
Therefore A is a T_2'-space.

Theorem 5. 2. *An open subset of a T_1''-space (T_0', T_1'-space) is, with the relative topology, a T_1''(T_0', T_1')-space.*

The proof of this theorem becomes direct after having the following lemma.

Lemma 5. 3. *Let (X, τ) be a topological space, (Y, τ') be an open subspace, and let $A \subset X$ be an open subset. If $y \in Y$ is a τ-boundary point of A, then y is a τ'-boundary point of $A \cap Y$.*

PROOF. If $y \in Y$ is a τ-boundary point of A, then every τ-neighborhood of y has a non-empty intersection with A. Since $Y \subset X$ is open in X, every τ'-neighborhood of y is a τ-neighborhood. Hence, every τ-neighborhood of y intersects $A \cap Y$. This implies that y is a τ'-boundary point of $A \cap Y$.

PROOF OF THEOREM 5. 2. Let (X, τ) be a T_1''-space and let (Y, τ') be any open subspace. For any two distinct points $y_1, y_2 \in Y$ there exists two τ-open neighborhoods V_1 and V_2 of y_1 and y_2 respectively, such that y_1 is a τ-boundary point of V_2 and y_2 is a τ-boundary point of V_1. The intersections of V_1 and V_2 with Y: $W_1 = V_1 \cap Y$ and $W_2 = V_2 \cap Y$, are two τ'-open neighborhoods of y_1 and y_2 respectively. By the above lemma y_1 is a τ'-boundary point of W_2 and y_2 is a τ'-boundary point of W_1. Hence (Y, τ') is a T_1''-space.

Theorem 5. 4. *A T_0'-space X which is also a T_1-space is a T_1'-space.*

PROOF. Let $x, y \in X$ be any two distanct points. Suppose that one of these points, say x, has an open neighborhood U and y is a boundary point of it. Since X is a T_1-space, one may consider x as a closed subset of X, and so $V = X - \{x\}$ is an open neighborhood of y which does not contain x. Hence X is a T_1'-space.

Theorem 5. 5. *A connected T_1-space is T_1''-space.*

PROOF. Let X be a connected T_1-space and let $x, y \in X$ be any two distinct points. Then $X - \{y\}$ and $X - \{x\}$ may be considered as two open neighborhoods of x and y respectively. Since X is connected,

$$\overline{X - \{x\}} = \overline{X - \{y\}} = X.$$

So x is a boundary point of $X - \{x\}$ and y is a boundary point of $X - \{y\}$. Hence X is a T_1''-space.

[5]) $\sim$ is the closure operation with respect to the relative topology.

We give now an example to show that the converse of this theorem is not necessarily true.

Example: The rationals, with the usual topology of the reals relativized is a T_1''-space but it is not a connected space.

Remark. Of course, if X is infinite, and τ is the smallest topology such that (X, τ) is a T_1-space, then (X, τ) is connected and therefore (X, τ) is a T_1''-space.

Theorem 5. 6. *A T_4-space is*[6]) T_2'.

PROOF. Let X be a T_4-space and let $x, y \in X$ be any two distinct points.

Since X is a T_1-space, we may consider x and y as two disjoint closed sets in X. As X is assumed to be normal, there are two disjoint open sets U, V such that $x \in U$ and $y \in V$. Hence, we get two disjoint open sets U_1 and V_1 (see [2]) such that $x \in U_1$, $y \in V_1$ and $\overline{U} \subset U$, $\overline{V}_1 \subset V$. So $\overline{U} \cap \overline{V}_1 = \emptyset$ and consequently X is T_2'-space.

Note: More interesting properties of these new spaces will be discussed in a coming paper.

References

[1] J. L. KELLEY, General Topology, *New York* (1955).
[2] E. M. PATTERSON, Topology, *London* (1956).

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[6]) A T_4-space is one which is normal and T_1 (cf. [1]).