

Two isotopically equivalent varieties of groupoids

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We shall say that a groupoid $\langle A, \omega' \rangle$ is nomially derived from a groupoid $\langle A, \omega \rangle$ if there exists a ω -word $w(x, y)$ in x, y such that $xy\omega' = w(x, y)$ for all $x, y \in A$. If further $\langle A, \omega \rangle$ is also nomially derived from $\langle A, \omega' \rangle$ we shall say that $\langle A, \omega \rangle, \langle A, \omega' \rangle$ are nomially equivalent. Groupoids $\langle A, \omega \rangle, \langle A, \omega' \rangle$ will be called isotopically equivalent if each is an isotope ([1]) of the other.

Two classes K_1, K_2 of groupoids will be called nomially (isotopically) equivalent if every groupoid in one class is nomially (isotopically) equivalent to a groupoid in the other.

Nomial and isotopical equivalences can be defined for arbitrary classes of universal algebras of a given species without any difficulty.

Examples of nomially equivalent varieties of algebras are well-known. Thus varieties of Boolean algebras and Boolean rings are nomially equivalent.

In this note we prove that the variety of groupoids defined by the law $y = xt\omega xz\omega yz\omega\omega t\omega\omega$ is isotopically equivalent to the variety of abelian groups.

Theorem. *The varieties of groupoids defined by the laws*

$$(I) \quad y = xt\omega xz\omega yz\omega\omega t\omega\omega$$

and

$$(II) \quad y = x xz\omega yz\omega\omega\omega$$

are isotopically equivalent but not nomially equivalent.

The PROOF of the theorem is divided into the following four lemmas.

Lemma 1. *A groupoid satisfying I is a quasigroup.*

PROOF. In I replace t by $y_2z\omega$, y by y_1 and x by $xz\omega$. We get the law

$$y_1 = xz\omega y_2z\omega\omega xz\omega z\omega y_1z\omega\omega y_2z\omega\omega\omega.$$

If we write w for the word $xz\omega z\omega y_1z\omega\omega y_2z\omega\omega$ and u for the word $xw\omega$ then this last law together with I gives

$$y_2 = xw\omega xz\omega y_2z\omega\omega w\omega = xw\omega y_1\omega = uy_1\omega.$$

Hence I implies that $\forall y_1, y_2 \exists t (ty_1\omega = y_2)$. This can be expressed by saying that the equation $ty_1\omega = y_2$ has a solution in t in every groupoid satisfying I for all y_1, y_2 .

The equation $y_1 t \omega = y_2$ is also solvable. This can be seen as follows. We have just proved that we can find x, t_1 such that $x t_1 \omega = y_1$. Then, by I,

$$y_1 x z \omega y_2 z \omega \omega t_1 \omega \omega = y_2$$

and if we put $t = x z \omega y_2 z \omega \omega t_1 \omega$ we have $y_1 t \omega = y_2$.

We now prove the 'cancellation laws'. Let t_1, y, t_2 be such that $t_1 y \omega = t_2 y \omega$. Then, by (I),

$$t_1 = x t \omega x y \omega t_1 y \omega \omega t \omega \omega = x t \omega x y \omega t_2 y \omega \omega t \omega \omega = t_2$$

and the right cancellation law holds. Next let t_1, y, t_2 be such that $y t_1 \omega = y t_2 \omega$. Find x, z, y_1, y_2 such that $y = x z \omega, t_1 = y_1 z \omega, t_2 = y_2 z \omega$. Then

$$\begin{aligned} y_1 &= x t \omega x z \omega y_1 z \omega \omega t \omega \omega = x t \omega y t_1 \omega t \omega \omega = x t \omega y t_2 \omega t \omega \omega = \\ &= x t \omega x z \omega y_2 z \omega \omega t \omega \omega y_2 \end{aligned}$$

so that $t_1 = t_2$ and the left cancellation law also holds. This completes the proof of the lemma.

Lemma 2. *The law (I) is equivalent to the following statement. There exists e such that for all x, y, z*

- (i) $x e \omega e \omega = x$,
- (ii) $x z \omega y z \omega \omega = x y \omega e \omega$,
- (iii) $x e \omega x y \omega \omega = y$.

PROOF. I implies (i)–(iii). We have, by (I)

$$x = x t \omega x z \omega x z \omega \omega t \omega \omega.$$

By Lemma 1 $x z \omega$ can be any element t_1 for suitable z . Hence

$$(iv) \quad x = x t_1 \omega t t_1 \omega t \omega \omega, \quad \text{for all } x, t_1, t.$$

This gives

$$x x \omega = x x \omega t \omega x x \omega t \omega \omega$$

and if t is such that $x x \omega t \omega = y$,

$$x x \omega = y y \omega = e, \quad \text{some fixed element.}$$

We can now write (iv) in the form

$$x = x t \omega e t \omega \omega.$$

In the last equation if t is taken equal to e we arrive at (i).

To prove (ii) we note that

$$y = x t \omega x z_1 \omega y z_1 \omega \omega t \omega \omega = x t \omega x z_2 \omega y z_2 \omega \omega t \omega \omega.$$

Hence by Lemma 1

$$x z_1 \omega y z_1 \omega \omega = x z_2 \omega y z_2 \omega \omega$$

so that $xz\omega yz\omega\omega = xy\omega\omega y\omega\omega = xy\omega e\omega$.

By (I) and (i)

$$y = x\omega\omega xy\omega y\omega\omega e\omega\omega = x\omega\omega xy\omega\omega.$$

This proves (iii).

Now we show that (i)–(iii) imply (I). We have

$$\begin{aligned} xt\omega yz\omega yz\omega\omega t\omega\omega &= xt\omega xy\omega e\omega t\omega\omega && \text{, by (ii) of Lemma 2} \\ &= x\omega\omega xy\omega e\omega e\omega\omega && \text{, by (ii) of Lemma 2} \\ &= x\omega\omega xy\omega\omega && \text{, by (i) of Lemma 2} \\ &= y && \text{, by (iii) of Lemma 2} \end{aligned}$$

Hence I holds and the lemma is proved.

Lemma 3. *A groupoid $\langle A, \omega \rangle$ satisfies (I) if and only if $\langle A, \omega' \rangle$ satisfies (II) where*

$$(1) \quad xy\omega = xy\omega'\gamma, \quad \text{for all } x, y \in A,$$

where $xy\omega' = xy\omega xx\omega\omega$ for all $x, y \in A$ and γ is an involution of $\langle A, \omega' \rangle$. (We recall that γ is called an involution if γ is an automorphism and γ^2 is the identity map.) Further a groupoid $\langle A, \omega' \rangle$ satisfies (II) if and only if $\langle A, \omega \rangle$ satisfies (I) where

$$(2) \quad xy\omega' = xy\omega xx\omega\omega \quad \text{for all } x, y \in A.$$

PROOF. *Part 1.* Let $\langle A, \omega' \rangle$ satisfy (II) and $\langle A, \omega \rangle$ be defined by (1). Then $\langle A, \omega \rangle$ satisfies (I). For,

$$\begin{aligned} xt\omega xz\omega yz\omega\omega t\omega\omega &= xt\omega' \gamma xz\omega' \gamma yz\omega' \gamma \omega' \gamma t\omega' \gamma \omega' \gamma, && \text{by (1)} \\ &= xt\omega' xz\omega' yz\omega' \omega' t\omega' \omega', && \text{since } \gamma \text{ is an involution} \\ &= y. \end{aligned}$$

In this last step we have used the result [2] that II characterizes abelian groups in terms of the operation of subtraction.

Part 2. Let $\langle A, \omega \rangle$ satisfy (I) and ω' be defined by (2). Then $\langle A, \omega' \rangle$ satisfies (II). For by the proof of Lemma 2 $xx\omega = e$ for some e and for all x in A and hence

$$\begin{aligned} xxz\omega' yz\omega'\omega'\omega' &= xxz\omega e\omega yz\omega e\omega\omega e\omega\omega e\omega && \text{, by (2)} \\ &= x xz\omega yz\omega\omega e\omega e\omega\omega e\omega && \text{, by (ii) of Lemma 2} \\ &= x xz\omega yz\omega\omega e\omega\omega && \text{, by (i) of Lemma 2} \\ &= x\omega\omega xz\omega yz\omega\omega e\omega\omega && \text{, by (ii) of Lemma 2} \\ &= y && \text{, by I.} \end{aligned}$$

Part 3. Let $\langle A, \omega \rangle$ satisfy (I) and let $\langle A, \omega' \rangle$ be defined by (2). We show that (I) holds for some involution γ of $\langle A, \omega' \rangle$. Define γ by $x\gamma = xe\omega$ where e is the constant value of $xx\omega$ in $\langle A, \omega \rangle$. Then γ is an involution of $\langle A, \omega' \rangle$. For

$$\begin{aligned} xy\omega'\gamma &= xy\omega e\omega e\omega \\ &= xe\omega ye\omega \omega e\omega, & \text{by (ii) of Lemma 2} \\ &= x\gamma y \gamma \omega e \omega = x\gamma y \gamma \omega' \end{aligned}$$

and

$$x\gamma\gamma = xe\omega e\omega = x, \quad \text{by (i) of Lemma 2.}$$

The involution γ satisfies (I); $xy\omega'\gamma = xy\omega e\omega e\omega = xy\omega$, by (i) of Lemma 2.

Part 4. Let $\langle A, \omega' \rangle$ satisfy II and $\langle A, \omega \rangle$ be defined (1). We show that ω, ω' are related by (2). This is easy:

$$xy\omega xx\omega\omega = xy\omega'\gamma xx\omega'\gamma \omega'\gamma = yx\omega' xx\omega' \omega' = xy\omega'.$$

Here we have used the fact that ω' is an operation of subtraction in an abelian group and the assumption that γ is an involution of $\langle A, \omega' \rangle$.

The proof of the lemma is complete.

We observe that Part 3 of the proof of Lemma 3 shows that if $\langle A, \omega \rangle$ satisfies I and ω' is defined by (2) then $\langle A, \omega' \rangle$ is an isotope of $\langle A, \omega \rangle$. In view of this remark Lemma 3 shows that the varieties defined by I and II are isotopically equivalent.

Lemma 4. *The varieties defined by (I) and (II) are not nomially equivalent.*

PROOF. Suppose that the lemma is not true. Let $\langle A, \omega \rangle$ be a countably free groupoid satisfying (I) and let $\langle A, \omega' \rangle$ be nomially equivalent to $\langle A, \omega \rangle$ and suppose that $\langle A, \omega' \rangle$ satisfies (II). Since endomorphisms of $\langle A, \omega \rangle$ and $\langle A, \omega' \rangle$ are the same it is easy to see that $\langle A, \omega' \rangle$ is a free abelian group. Let k_1, k_2 be integers such that $xy\omega = k_1x + k_2y$ for all $x, y \in A$. Here we have used the usual operation of addition in an abelian group and usual notation. If we substitute $xy\omega = k_1x + k_2y$ in (I) and use the fact that $\langle A, \omega' \rangle$ is free we find that $k_1k_2 = \pm 1$. This implies that $\langle A, \omega \rangle$ satisfies (II) or the law

$$(II^*) \quad y = zy\omega zx\omega \omega x\omega.$$

But $\langle A, \omega \rangle$ is free. Hence (I) implies (II) or (II*).

However there exist groupoids that satisfy (I) but not (II) or (II*). An example is the groupoid consisting of four elements 0, 1, 2, 3 in which the multiplication is defined by the table:

	0	1	2	3
0	0	3	2	1
1	3	0	1	2
2	2	1	0	3
3	1	2	3	0

This contradiction proves the lemma.

Incidentally the law (I) has the interesting property that if $\langle A, \omega \rangle$ satisfies (I) then so does the dual groupoid $\langle A, \omega^* \rangle$ where $xy\omega = yx\omega^*$ for all $x, y \in A$. The law (II) does not have this property.

In the end I would like to thank Professor B. H. NEUMANN for his encouragement during the writing of this paper.

References

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(Received June 20, 1968.)