

## Representation of integers by norm forms II.

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### 1. Introduction

Let  $1, \alpha_2, \dots, \alpha_m$  be linearly independent algebraic numbers over the field  $R$  of rationals, and let  $n$  denote the degree of the algebraic number field  $K = R(\alpha_1, \dots, \alpha_m)$  (over  $R$ ). The conjugates and the norm of an element  $\alpha \in K$  will be denoted by  $\alpha^{(1)}, \dots, \alpha^{(n)}$  and  $N_{K/R}(\alpha)$ , respectively (in the field  $K$ ). Let further

$$L^{(k)}(\mathbf{x}) = x_1 + \alpha_2^{(k)} x_2 + \dots + \alpha_m^{(k)} x_m \quad (k = 1, \dots, n)$$

It is a question examined by many authors when the Diophantine equation

$$(1) \quad N_{K/R}(L(\mathbf{x})) = a; \quad a \in R$$

has infinitely many solutions  $\mathbf{x} = (x_1, \dots, x_m)$  among rational integers.

In case  $m = n$  the problem is solved. If (1) has a solution then it has infinitely many solutions and these can be represented by help of the units of  $K$  ([2] pp. 134—140.).

In case  $m \leq n$  the question is much more difficult, the answer depends on the structure of the module  $M = \{1, \alpha_2, \dots, \alpha_m\}$ . The module  $M$  is called *degenerated*, if the vector space  $L$  over  $R$  generated by  $M$  has a subspace  $L'$  such that, for some  $\gamma \in K$ ,  $L'\gamma$  is a (not necessarily proper) subfield of  $K$ , different from  $R$  and the imaginary number fields of degree 2. Now if  $M$  is degenerated then there exists an  $a \in R$  such that (1) has infinitely many solutions ([2], p. 322.). In the opposite case — i.e. if  $M$  is non-degenerated — by conjecture (1) has only finitely many solutions among rational integers for any  $a \in R$ .

This conjecture was proved by A. THUE ([5]) for  $m = 2$  and by W. M. SCHMIDT ([4]) for  $m = 3$ . However, their method is not effective, it is not suitable to find all the solutions of (1). Recently A. BAKER ([1]) has given an algorithm to find all solutions of (1) in case  $m = 2$  by showing that if  $\varkappa > n + 1$  then every solution of (1) satisfies

$$\max(|x_1|, |x_2|) < \exp \{n^{\varkappa} H^{\varkappa} + (\log |a| \cdot H)^n\} = \varphi(a, n, H, \varkappa)$$

where  $H = H(\alpha_2)$  is the height<sup>1)</sup> of  $\alpha_2$  and  $v = 32\varkappa/(\varkappa - u - 1)$  (supposing naturally that  $M$  is non-degenerated).

<sup>1)</sup> The maximum of absolute values of the relatively prime integer coefficients in the defining polynomial of  $\alpha_2$ . In knowledge of  $N(x_1 + \alpha_2 x_2)$ ,  $H(\alpha_2)$  can be considered to be known too.

In [3] one of the authors proved the conjecture for  $m=4$  and non-real Abelian number fields of degree not divisible by 3 and 4.

In the present paper we investigate the problem for those number fields  $K$  which are contained in a Galois field  $F$  (a normal extension of  $R$ ) whose maximal real subfield is also a Galois field. These fields we be called *allowed*; thus allowed fields are the subfields of the non-real normal extensions (over  $R$ ) of degree 2 of the real Galois fields, E.g. all Abelian number fields are allowed. We are going to show that in the case of allowed number fields  $K$  any solution  $\mathbf{x} = (x_1, \dots, x_m)$  of (1) satisfies both

$$(2) \quad |N_{F/R}(\operatorname{Re} L(\mathbf{x}))| \leq |a|^{[F:K]} \quad \text{and} \quad |N_{F/R}(i \operatorname{Im} L(\mathbf{x}))| \leq |a|^{[F:K]}$$

Using this and the above mentioned result of Baker we give an explicit bound for  $\max(|x_1|, |x_2|, |x_3|)$  in case  $m=3$ , if  $K$  is a non-real allowed number field. Furthermore, we prove the conjecture <sup>2)</sup> in case  $m=4$  for non-real allowed Galois fields of degree not divisible by 3 and 4. We give a second proof of this latter proposition too, using Dirichlet's theorem concerning units of algebraic number fields instead of (2).

## 2. Results

To state our results we need the following constants. Let

$$\begin{aligned} c(n, m, H) &= [(m-1)!^6 n^{14(m-1)^2} H^{6m(m-1)^2}]^{n^{5m(m-1)^2}}, \\ H_1(n, m, H) &= [(m-1)c(n, m, H)n^{m-1} H^m]^{n^{m-1}}, \\ b(a, n, H) &= (2H_1(n, 3, H)^3)^{n^2} |a|^n c(n, 3, H)^{4n}, \\ H_2(n, H) &= [2H_1(n, 3, H)]^{5n^2}, \quad H_3(n, H) = [4n^4 H_2(n, H)]^{n^4}, \\ \psi(a, n, H, \varkappa) &= b^2 H_3^2 \varphi(b^{n^4}, n^3, H_3, \varkappa). \end{aligned}$$

Then we have

**Theorem 1.** *Let  $\{1, \alpha_2, \alpha_3\}$  be a non-degenerated module with linearly independent generators such that the field  $K = R(\alpha_2, \alpha_3)$  is a non-real allowed number field of degree  $n$ . If the height of  $\alpha_1$  and  $\alpha_2$  is  $\leq H$  and  $\varkappa > n^6 + 1$ , then any solution  $(x_1, x_2, x_3)$  of the equation*

$$(1') \quad N_{K/R}(x_1 + \alpha_2 x_2 + \alpha_3 x_3) = a \quad (a \in R)$$

satisfies

$$\max(|x_1|, |x_2|, |x_3|) \leq \psi(a, n, H, \varkappa)$$

**Theorem 2.** *Let  $F$  be a non-real allowed Galois field of degree not divisible by 3 and 4. Let further  $1, \alpha_2, \alpha_3, \alpha_4$  be linearly independent generators of  $F$ . Then the equation*

$$(1'') \quad N_{F/R}(x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4) = a \quad (a \in R)$$

has only finitely many solutions  $(x_1, x_2, x_3, x_4)$  in rational integers.

<sup>2)</sup> Added in proof: Recently Prof W. Schmidt proved the conjecture for all  $m$ .

### 3. Lemmas

**Lemma 1.** *If the coefficients of the linear form  $L(\mathbf{x})$  are linearly independent elements of an allowed field  $K$  and  $F \supseteq K$  is an allowed Galois-field, then any solution of (1) satisfies (2).*

**PROOF.** If  $K$  is real we have nothing to prove; assume that  $K$  and consequently  $F$  are non-real. Let  $G$  be the Galois group of  $F$  (over  $R$ ). Since in our assumption the maximal real subfield  $S$  of  $F$  is normal, the cyclic group  $\{\psi\}$  (of order 2) generated by the complex conjugation  $\psi$  (as automorphism) is a normal subgroup. Hence  $\psi$  belongs to the center of  $G$ .

Now if  $H$  is the subgroup corresponding to  $K$  then all isomorphisms of  $K$  in  $F$  can be described by the right cosets of  $H$ . Let  $H\varphi_1, \dots, H\varphi_n$  be these cosets. Since  $\psi \in Z(G)$ , we have for any  $\varphi \in H\varphi_k$  ( $1 \leq k \leq n$ )

$$(\operatorname{Re} L(\mathbf{x}))\varphi = \frac{1}{2}(L(\mathbf{x}) + L(\mathbf{x})\psi)\varphi = \frac{1}{2}(L(\mathbf{x})\varphi_k + L(\mathbf{x})\varphi_k\psi) = \operatorname{Re}(L(\mathbf{x})\varphi_k)$$

Hence

$$N_{F/R}(\operatorname{Re} L(\mathbf{x})) = \prod_{\varphi} (\operatorname{Re} L(\mathbf{x}))\varphi = \left[ \prod_{k=1}^n \operatorname{Re}(L(\mathbf{x})\varphi_k) \right]^{[F:K]}$$

and since

$$|L(\mathbf{x})\varphi_k| \cong |\operatorname{Re}(L(\mathbf{x})\varphi_k)|$$

we have

$$|a|^{[F:K]} = |N_{F/R}(L(\mathbf{x}))| \cong \left| \prod_{k=1}^n \operatorname{Re}(L(\mathbf{x})\varphi_k) \right|^{[F:K]} = |N_{F/R}(\operatorname{Re} L(\mathbf{x}))|$$

The second statement of (2) follows similarly.

We are going to apply Baker's result onto one of the inequalities (2). This can be done only if one of the forms  $\operatorname{Re} L(\mathbf{x})$ ,  $\operatorname{Im} L(\mathbf{x})$  has two variables at the most, its coefficients are linearly independent and generate a non-generated module. This is not the case in general but it can be reached by multiplying (1') by the norm of an appropriate element of  $F$  and by applying a linear transformation. In lemmas 2. and 3. this will be carried out.

Before stating them we remark that the heights of  $\alpha$ ,  $\bar{\alpha}$  and  $1/\alpha$  are equal and the same holds for their degrees. Furthermore, if  $d(\alpha)$  is the degree of  $\alpha$  then  $|\alpha| \cong d(\alpha)H(\alpha)$ . Using this one easily verifies that  $d(\alpha\beta)$ ,  $d(\alpha + \beta) \cong d(\alpha)d(\beta)$ , and if  $f$  is a polynomial of degree  $l$  having integer coefficients and  $s \cong l$  variables,  $H(\alpha_i) \cong H$  and  $d(\alpha_i) \cong d$  ( $1 \leq i \leq s$ ), then

$$d(f(\alpha_1, \dots, \alpha_s)) \cong d^s,$$

$$(3) \quad H(f(\alpha_1, \dots, \alpha_s)) \cong (\|f\| \cdot d^s H^{l(s+1)})^{d^s}$$

(here  $\|f\|$  means the sum of absolute values of the coefficients of  $f$ ).

**Lemma 2.** *One can find a linear, non-singular transformation  $Y = CX$  with integer coefficients transforming (1) into an equation*

$$(4) \quad N_{K/R}(y_1 + \alpha'_2 y_2 + \dots + \alpha'_m y_m) = a_1$$



Now we deal with the necessary estimations. First let us consider the heights of the  $c'_{r,s}$ -s. Since in our assumption  $i \operatorname{Im} \alpha_2, \dots, i \operatorname{Im} \alpha_l$  are linearly independent, we can find certain indices  $i_1, \dots, i_{l-1}$  such that

$$\beta = \begin{vmatrix} (i \operatorname{Im} \alpha_2)^{(i_1)} & \dots & (i \operatorname{Im} \alpha_l)^{(i_1)} \\ \vdots & & \vdots \\ (i \operatorname{Im} \alpha_2)^{(i_{l-1})} & \dots & (i \operatorname{Im} \alpha_l)^{(i_{l-1})} \end{vmatrix}$$

Then  $c'_{r,s}$  ( $l+1 \leq r \leq m, 2 \leq s \leq l$ ) can be determined from the system of equalities

$$\begin{aligned} (i \operatorname{Im} \alpha_r)^{(i_1)} &= c'_{r,2} (i \operatorname{Im} \alpha_2)^{(i_1)} + \dots + c'_{r,l} (i \operatorname{Im} \alpha_l)^{(i_1)} \\ &\vdots \\ (i \operatorname{Im} \alpha_r)^{(i_{l-1})} &= c'_{r,2} (i \operatorname{Im} \alpha_2)^{(i_{l-1})} + \dots + c'_{r,l} (i \operatorname{Im} \alpha_l)^{(i_{l-1})} \end{aligned}$$

as a quotient where the denominator is the determinant  $\beta$  and the numerator is a determinant of similar type. Now by (3),

$$d(i \operatorname{Im} \alpha_k) = d\left(\frac{\alpha_k - \bar{\alpha}_k}{2}\right) \leq n^2, \quad H(i \operatorname{Im} \alpha_k) \leq (4n^2 H^3)^{n^2},$$

$$d\left(\frac{1}{\beta}\right) = d(\beta) \leq n^{2(m-1)^2}, \quad H\left(\frac{1}{\beta}\right) = H(\beta) \leq [(m-1)! n^{2(m-1)^2} H^{m(m-1)^2}]^{n^{(m-1)^2}}$$

hence, applying (3) again,

$$H(c'_{r,s}) \leq c(n, m, H).$$

$c$  being the l. c. m. of the denominators of the  $c'_{r,s}$ -s,

$$\max(|c|, |c_{r,s}|) \leq c(n, m, H)^{(m-1)^2}$$

Hence

$$|a_1| = |ac^n| \leq |a| \cdot c(n, m, H)^{(m-1)^2 n}$$

Since

$$|c'_{r,s}| \leq H(c'_{r,s}) \leq c(n, m, H),$$

(6) gives

$$\max_i |x_i| \leq (m-1) c(n, m, H) \max_i |y_i|$$

and, finally, (7) implies

$$H(\alpha'_i) \leq H_1(n, m, H),$$

which was to be proved.

**Lemma 3.** *Let  $1, \alpha_2, \alpha_3$  be linearly independent algebraic numbers and assume that  $M = \{1, \alpha_2, \alpha_3\}$  is non-real and non-degenerated. Then there exists a linear, non-singular transformation  $Z = AX$  with integer coefficients, such that the application of  $A$  and appropriate exponentation of both sides of (1') transforms (1') into an equation*

$$(8) \quad N_{K'/R}(\alpha'_1 z_1 + \alpha'_2 z_2 + \alpha'_3 z_3) = a_2$$

where  $K \subseteq K' = R(\alpha'_1, \alpha'_2, \alpha'_3)$  is of degree  $\leq n^2$ , the module  $\{i \operatorname{Im} \alpha'_1, i \operatorname{Im} \alpha'_2, i \operatorname{Im} \alpha'_3\}$

has at most two generators different from 0, these are linearly independent and generate a non-degenerated module, furthermore

$$|a_2| \leq |a|^n c(n, 3, H)^{4n}$$

$$H(\alpha'_i) \leq H_1(n, 3, H)$$

and the corresponding solutions of (1') and (8) satisfy

$$\max_i |x_i| \leq 2c(n, 3, H) \max_i |y_i|$$

*Remark:* The proof shows that if  $K$  is allowed then so is  $K'$ .

**PROOF.** Put  $\alpha_k = \beta_k + i\gamma_k$  ( $k=2, 3$ ) in (1'). If  $\gamma_2$  and  $\gamma_3$  are linearly dependent then by lemma 2 (1') can be transformed into a form like (8), where  $K' = K$  and then the required inequalities follow trivially.

Thus we may assume that  $\gamma_2, \gamma_3$  are linearly independent. Put  $i\gamma_3 = \gamma \cdot i\gamma_2$ . If  $\gamma$  is of degree  $\geq 3$  we have nothing to prove since (1') itself satisfies the requirements. Thus we may confine ourselves to the case  $d(\gamma) = 2$ .

Put  $\delta = \beta_3 - \gamma\beta_2 = \alpha_3 - \gamma\alpha_2$ . Now  $\delta \in R(\gamma)$ , since otherwise we would have  $\alpha_3 - \gamma\alpha_2 = r_1\gamma - r_2$  with some rational coefficients  $r_1, r_2$ , which would imply

$$\gamma = \frac{\alpha_3 - r_2}{\alpha_2 + r_1},$$

i.e. that  $\{1, \alpha_2, \alpha_3\}$  is degenerated, which is a contradiction. Hence either  $\gamma - \delta$  or  $\gamma + \delta$ , say  $\gamma - \delta$  is of degree  $\geq 3$ .

Put  $K' = K(\bar{\alpha}_2) \subseteq F$ . Consider the  $[K':K]$ -th power of both sides of (1'), multiply by  $N_{K'/R}(1 + \bar{\alpha}_2)$  and substitute  $z_1 = x_1 - x_2, z_2 = x_2, z_3 = x_3$

$$N_{K'/R}((1 + \bar{\alpha}_2)z_1 + (1 + \bar{\alpha}_2)(1 + \alpha_2)z_2 + (1 + \bar{\alpha}_2)\alpha_3z_3) = a^{[K':K]} N_{K'/R}(1 + \bar{\alpha}_2).$$

Here  $\text{Im}(1 + \bar{\alpha}_2) = \gamma_2, \text{Im}(1 + \bar{\alpha}_2)(1 + \alpha_2) = 0, \text{Im}(1 + \bar{\alpha}_2)\alpha_3 = \gamma_2(\gamma - \delta)$ , i.e. we have obtained an equality of the required type. The necessary estimations are trivial.

#### 4. Proofs of the theorems

*Proof of theorem 1.* By lemma 3, it is enough to find an upper bound for the solutions of equality (8). We may assume that  $\text{Im } \alpha'_1 = 0$  and either  $\text{Im } \alpha'_2 = 0, \text{Im } \alpha'_3 \neq 0$  or  $\text{Im } \alpha'_2, \text{Im } \alpha'_3$  are linearly independent and  $\{i \text{Im } \alpha'_2, i \text{Im } \alpha'_3\}$  is non-degenerated.

There exists a rational integer  $c$  with  $|c| \leq 2H_2^3$  such that  $\alpha''_k = c\alpha'_k, i \text{Im } \alpha''_k$  ( $k=1, 2, 3$ ) are algebraic integers. Multiplying both sides of (8) by  $N_{K'/R}(c) = c^{[K':R]}$ ,

$$(9) \quad N_{K'/R}(\alpha''_1 z_1 + \alpha''_2 z_2 + \alpha''_3 z_3) = a_3$$

where  $|a_3| = |a_2 c^{[K':R]}| \leq b(a, n, H)$  and  $H(\alpha''_k) \leq H_2(n, H)$ .

Assume first that in (8) we have  $\text{Im } \alpha'_1 = \text{Im } \alpha'_2 = 0$ . Let  $F$  be an allowed Galois extension of  $K$ , then, as we have noticed after lemma 3,  $K' \subseteq F$  and

similarly  $K'' = K'(\bar{\alpha}_3'') \subseteq F$  where  $K''$  is of degree  $\leq n^4$ . Applying now lemma 1 onto the equality (9), we obtain

$$|a_3|^{[F:K']} \cong |N_{K''/R}(i \operatorname{Im} \alpha_3'' z_3)|^{[F:K']} \cong |z_3|^{[F:R]},$$

hence

$$|z_3| \cong |a_3| \cong b(a, n, H).$$

Now if  $z_3 = 0$ , then (9) gives, by the mentioned result of Baker, that

$$\max(|z_1|, |z_2|) \cong \varphi(b, n^2, H_2, \varkappa).$$

If  $z_3 \neq 0$ , (9) gives

$$|a_3|^{[F:K']} \cong \left| \frac{L^{(k)}(z)}{i \operatorname{Im} L^{(k)}(z)} \right|^{[F:K']} |N_{K''/R}(i \operatorname{Im} \alpha_3'' z_3)|^{[F:K']} \cong \left| \frac{L^{(k)}(z)}{i \operatorname{Im} \alpha_3''^{(k)}} \right|^{[F:K']}$$

hence we have for every  $1 \leq k \leq n$

$$|L^{(k)}(z)| \cong |a_3| |i \operatorname{Im} \alpha_3''^{(k)}| \cong b n^4 H_3$$

Now  $\alpha_1'', \alpha_2'', \alpha_3''$  are linearly independent, therefore we can find a pair  $j, k$  of indices such that

$$\beta = \begin{vmatrix} \alpha_1''^{(j)} & \alpha_2''^{(j)} \\ \alpha_1''^{(k)} & \alpha_2''^{(k)} \end{vmatrix} \neq 0$$

Expressing  $z_1, z_2$  from the equalities

$$\begin{aligned} \alpha_1''^{(j)} z_1 + \alpha_2''^{(j)} z_2 &= L^{(j)}(z) - \alpha_3''^{(j)} z_3 \\ \alpha_1''^{(k)} z_1 + \alpha_2''^{(k)} z_2 &= L^{(k)}(z) - \alpha_3''^{(k)} z_3 \end{aligned}$$

we can find an upper bound for them with the help of an upper bound for the height of  $\beta$ , which can be obtained similarly as in the proof of lemma 2. The upper bound for  $|x_1|, |x_2|, |x_3|$  deduced from this is much better than the upper bound given in the theorem.

Assume now that in (8)  $\operatorname{Im} \alpha_1'' = 0$ ,  $\operatorname{Im} \alpha_2'', \operatorname{Im} \alpha_3''$  are linearly independent and  $\{i \operatorname{Im} \alpha_2'', i \operatorname{Im} \alpha_3''\}$  is non-degenerated. Consider the field  $K''' = K'(\bar{\alpha}_2'', \bar{\alpha}_3'') \subset F$  of degree  $\leq n^6$  and apply lemma 1. Then (9) gives the inequality

$$|a_3|^{n^4} \cong |a_3|^{[F:K']/[F:K''']} \cong |N_{K'''/R}(i \operatorname{Im} \alpha_2'' z_2 + i \operatorname{Im} \alpha_3'' z_3)|$$

which implies by Baker

$$\max(|z_1|, |z_2|) \cong \varphi(b^{n^4}, n^3, H_3, \varkappa)$$

Now if  $z_2 = z_3 = 0$  then (9) gives

$$b \cong |a_3| = |N_{K'/R}(\alpha_1'' z_1)| \cong |z_1|$$

which is not larger than the upper bound given above. On the other hand, if one of  $z_2$  and  $z_3$  is non-zero, then by

$$|a_3|^{[F:K']} \cong \left| \frac{L^{(k)}(z)}{i \operatorname{Im} L^{(k)}(z)} \right|^{[F:K']} |N_{K'''/R}(i \operatorname{Im} \alpha_2'' z_2 + i \operatorname{Im} \alpha_3'' z_3)|^{[F:K''']} \cong \frac{|L^{(k)}(z)|^{[F:K']}}{|i \operatorname{Im} L^{(k)}(z)|^{[F:K']}}$$

and

$$\alpha_1'' z_1 = L^{(k)}(z) - \alpha_2'' z_2 - \alpha_3'' z_3$$

we can deduce

$$|z_1| \leq 4n^6 b H_2 H_3 \varphi(b^{n^4}, n^3, H_3, \varkappa)$$

which proves the theorem.

1ST PROOF OF THEOREM 2. By our assumption  $\alpha_2, \alpha_3, \alpha_4$  are not all real. Moreover, by lemma 2 we may suppose that the non-zero imaginary parts of  $\alpha_1, \alpha_2, \alpha_3$ , say  $\text{Im } \alpha_k, \dots, \text{Im } \alpha_4$ , are linearly independent. Obviously we may assume that they are algebraic integers. Now by lemma 1,

$$|a| \equiv |_{F/R}(i \text{Im } \alpha_k x_k + \dots + i \text{Im } \alpha_4 x_4)|$$

By our assumption concerning the degree of  $F$ ,  $F$  has no real subfield of degree 2 and no subfield of degree 3, hence the module  $\{i \text{Im } \alpha_k, \dots, i \text{Im } \alpha_4\}$  is non-degenerated. By Thue's and Schmidt's mentioned results this inequality has only finitely many solutions. Thus it is enough to show that for fixed  $x_k, \dots, x_4$  (1'') has only finitely many solutions in  $x_1, \dots, x_{k-1}$ .

If  $x_k = \dots = x_4 = 0$ , then (1'') gives

$$N_{F/R}(x_1 + \alpha_2 x_2 + \dots + \alpha_{k-1} x_{k-1}) = a$$

and here  $\{1, \alpha_2, \dots, \alpha_{k-1}\}$  is non-degenerated, consequently  $x_1, \dots, x_{k-1}$  can have only finitely many different values. On the other hand, if there is an  $x_j$  ( $k \leq j \leq 4$ ) different from 0, then because of the linear independence of  $\text{Im } \alpha_k, \dots, \text{Im } \alpha_4$ ,

$$|a| \equiv \left| \frac{L^{(r)}(\mathbf{x})}{i \text{Im } L^{(r)}(\mathbf{x})} \right| \cdot |N_{F/R}(i \text{Im } L(\mathbf{x}))| \equiv \frac{|L^{(r)}(\mathbf{x})|}{|i \text{Im } L^{(r)}(\mathbf{x})|} \quad (r = 1, \dots, [F:R])$$

and this gives an upper bound for  $L^{(r)}(\mathbf{x})$  independently from the value of  $x_1, \dots, x_{k-1}$ . Since  $1, \alpha_1, \dots, \alpha_{k-1}$  are linearly independent, there exist indices  $i_1, \dots, i_{k-1}$  such that

$$\begin{vmatrix} 1 & \alpha_2^{(i_1)} & \dots & \alpha_{k-1}^{(i_1)} \\ \vdots & & & \\ 1 & \alpha_2^{(i_{k-1})} & \dots & \alpha_{k-1}^{(i_{k-1})} \end{vmatrix} \neq 0$$

Hence the system

$$\begin{aligned} x_1 + \alpha_2^{(i_1)} x_2 + \dots + \alpha_{k-1}^{(i_1)} x_{k-1} &= L^{(i_1)}(\mathbf{x}) - \alpha_k^{(i_1)} x_k - \dots - \alpha_4^{(i_1)} x_4 \\ &\vdots \\ x_1 + \alpha_2^{(i_{k-1})} x_2 + \dots + \alpha_{k-1}^{(i_{k-1})} x_{k-1} &= L^{(i_{k-1})}(\mathbf{x}) - \alpha_k^{(i_{k-1})} x_k - \dots - \alpha_4^{(i_{k-1})} x_4 \end{aligned}$$

gives an upper bound for  $|x_1|, \dots, |x_{k-1}|$ .

2ND PROOF OF THEOREM 2. We start like in the 1st proof:  $\alpha_2, \alpha_3, \alpha_4$  are not all real,  $\text{Im } \alpha_1 = \dots = \text{Im } \alpha_{k-1} = 0$ , while  $\text{Im } \alpha_k, \dots, \text{Im } \alpha_4$  are linearly independent algebraic integers.



Assume indirectly that (1'') has infinitely many solutions. Since only finitely many principal ideals of  $F$  can have norm  $a$ , we can find an appropriate element  $\beta \in F$  such that infinitely many solutions of (1'') and units  $\varepsilon$  of  $F$  satisfy

$$x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \beta \varepsilon.$$

Since  $F$  is a non-real Galois field, it is totally imaginary, hence by Dirichlet's theorem the rank of the group of its units is  $\frac{1}{2}[F:R]-1$ . On the other hand,  $F$  being allowed its maximal real subfield is normal, consequently totally real, thus the rank of the group of the units of  $S$  is also  $\frac{1}{2}[F:R]-1$ . Now the logarithmical representation of the units shows that the units of  $S$  form a subgroup of the group of units of  $F$ , and the index of this subgroup is a finite number  $h$ . Therefore, there are units  $\varepsilon_1, \dots, \varepsilon_h$  such that every unity of  $F$  is of form  $\varepsilon_j \varepsilon$ , where  $\varepsilon$  is a real unity. Consequently we can find a fixed integer  $j$  such that

$$x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = \beta \varepsilon_j \varepsilon$$

has infinitely many solutions in integers  $x_1, x_2, x_3, x_4$  and real units  $\varepsilon$ . Considering the imaginary parts of both sides,

$$i \operatorname{Im} \alpha_k x_k + \dots + i \operatorname{Im} \alpha_4 x_4 = \gamma \varepsilon$$

where  $\gamma = \frac{1}{2}(\beta \varepsilon_j - \overline{\beta \varepsilon_j})$ . Taking the norm of both sides,

$$N_{F/R}(i \operatorname{Im} \alpha_k x_k + \dots + i \operatorname{Im} \alpha_4 x_4) = N_{F/R}(\gamma \varepsilon) = N_{F/R}(\gamma) = \text{const.}$$

which has, by our assumption concerning the degree of  $F$ , only finitely many solutions  $x_k, \dots, x_4$ . But then

$$x_1 + \alpha_2 x_2 + \dots + \alpha_{k-1} x_{k-1} = \beta \varepsilon_j \varepsilon - \alpha_k x_k - \dots - \alpha_4 x_4 = \eta$$

or

$$N_{F/R}(x_1 + \alpha_2 x_2 + \dots + \alpha_{k-1} x_{k-1}) = N_{F/R}(\eta)$$

has infinitely many solutions for some fixed  $x_k, \dots, x_4$  (and, consequently, for fixed  $\varepsilon$ ), which is a contradiction.

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