A note on almost-Dedekind domains

By ELBERT M. PIRTLE, JR. (Kansas City, Mo)

1. Introduction

In this section we state the necessary background results from [1]. In the remainder of this paper, R will denote a commutative integral domain with identity and quotient field K. $I(R)$ will denote the collection of non-zero fractionary ideals of R. A fractionary ideal of the form Rx, $x \neq 0$, $x \in K$, is called a principal fractionary ideal.

A relation $\prec$ is defined on $I(R)$ as follows: $A \prec B$ iff every principal fractionary ideal of R which contains A also contains B. The relation $\prec$ is reflexive and transitive. If we define $\equiv$ on $I(R)$ by $A \equiv B$ iff $A \prec B$, and $B \prec A$, for $A, B \in I(R)$, then $\equiv$ is an equivalence relation on $I(R)$. For $A \in I(R)$, $\operatorname{div}_R(A)$ denotes the equivalence class of A with respect to $\equiv$ and is called the divisor of A; $\mathscr{D}(R)$ denotes the set of all such equivalence classes.

For $A \in I(R)$, we put $\tilde{A} = \bigcap_{A \subseteq Rx} Rx$, $x \in K$, $x \neq 0$. A fractionary ideal B of R is said to be divisorial if $\tilde{B} = B$. It follows that for $A \in I(R)$, $\operatorname{div}_R(A) = \operatorname{div}_R(\tilde{A})$ and that $\tilde{A}$ is the unique divisorial fractionary ideal belonging to $\operatorname{div}_R(A)$. It also follows from the definition that $\widetilde{\tilde{A}\tilde{B}} = \widetilde{AB}$ for A, $B \in I(R)$, so that $\mathscr{D}(R)$, together with the operation $+$ defined by $\operatorname{div}_R(A) + \operatorname{div}_R(B) = \operatorname{div}_R(AB)$ is a commutative semigroup with identity $0 = \operatorname{div}_R(R)$. Furthermore, $\mathscr{D}(R)$ is a group iff R is completely integrally closed. If we define $\leqq$ on $\mathscr{D}(R)$ by $\operatorname{div}_R(A) \leqq \operatorname{div}_R(B)$ iff $A \prec B$, then $\mathscr{D}(R)$ is a lattice-ordered semigroup.

2. An application of $\mathscr{D}(R)$

It is shown in [1] that the proper prime ideals in a Dedekind domain are divisorial. We now apply the general theory of $\mathscr{D}(R)$ to show that this property characterizes. Dedekind domains in the class of one-dimensional Prüfer domains.

Theorem 2. 1. *Let R be a Prüfer domain in which proper prime ideals are maximal. Then R is a Dedekind domain iff every minimal pirme ideal of R is divisorial.*

Proof. ($\Leftarrow$) Since R is a one-dimensional Prüfer domain, R_M is a rank one valuation ring for each maximal ideal M of R. By [5, page 94], $R = \bigcap R_M$, where M runs over the collection of maximal ideals of R. Since R_M is completely integrally closed for each maximal ideal M of R, it follows that R is completely integrally closed, so that $\mathscr{D}(R)$ is a group (see [1]). Let P be any non-zero maximal (hence minimal) prime of R. Then $P(R:P) \subseteqq R$. If $P(R:P) \neq R$, then $P(R:P) \subseteqq M$ for some maximal ideal M of R. Then $P^2 \subseteqq P(R:P) \subseteqq M$; i.e., $P^2 \subseteqq M$. Thus $P \subseteqq M$, and so $P = M$ since both M and P are minimal (maximal). So we have $P(R:P) \subseteqq P$ if $P(R:P) \neq R$. But then in $\mathscr{D}(R)$ we have $0 = \operatorname{div}_R(P(R:P)) \geqq \operatorname{div}_R(P) > 0$, a contradiction ($\operatorname{div}_R(P) > 0$ since $P \subseteqq R$, and $\tilde{P} = P < R$). So we must have $P(R:P) = R$; i.e., P is invertible and hence finitely generated. Thus R is a ring in which every prime ideal is finitely generated and so by Cohen's theorem, [6, page 8] (see also [7]), R is Noetherian. We already have that R is integrally closed and that prime ideals are maximal. It follows that R is a Dedekind domain.

($\Rightarrow$) See [1].

In [4], Gilmer defines an almost-Dedekind domain (AD-domain) to be a domain R such that R_M is a Dedekind domain for each maximal ideal M of R. It follows that if R is an almost-Dedekind domain then R is a Prüfer domain in which prime ideals are maximal. We can now state the following corollary.

Corollary 2.2. Let R be an almost-Dedekind domain. Then R is Dedekind iff proper prime ideals of R are divisorial.

3. families of valuations and the construction of $\mathscr{A}(R)$

In this section we obtain a characterization of an AD-domain R in terms of a family F of valuations on K. The family is used to construct a partially ordered semigroup $\mathscr{A}(R)$ of fractionary ideal classes and the relation between $\mathscr{A}(R)$ and $\mathscr{D}(R)$ is described.

Definition 3.1. Let v be a rank one, discrete valuation on K which is non-negative on R. For any fractionary ideal A of R, put $v(A) = \inf\limits_{a \in A} v(A)$.

Theorem 3. 2. *R is AD iff there is a family F of valuations on K such that.*

(i) *Each $v \in F$ has rank one and is discrete.*
(ii) $R = \bigcap\limits_{v \in F} R_v$.
(iii) $R_v = R_{P(v)}$, *where $P(v)$ denotes the center of v on R for each $v \in F$.*
(iv) *R is the only ideal A of R such that $v(A) = 0$ for all $v \in F$.*

PROOF. Suppose F is a family of valuations on K which satisfies (i), (ii), (iii), (iv) above, and let P be any proper prime ideal of R. By (ii), $v(P) \geqq 0$ for all $v \in F$. By (iv) there is $v \in F$ such that $v(P) > 0$. Then $(0) < P \subseteqq P(v) < R$, where $P(v)$ denotes the center of v on R. Since $P(v)$ is the center of a rank one valuation, $P(v)$ is a minimal prime in R. Thus $P = P(v)$ and $R_P = R_{P(v)} = R_v$ is a rank one, discrete valuation ring. It follows that R is AD.

Suppose R is AD. Let F be the family of valuations on K induced by the family of proper primes of R. It is easy to see that F satisfies (i), (ii), (iii). To see that F satisfies (iv), let A be any proper ideal of R. Then $A \subseteqq M$ for some maximal (and hence minimal prime) ideal M of R. If v denotes the valuation on K induced by

M, then $v \in F$. Furthermore, $v(A) \geqq v(M) > 0$. Thus R must be the only ideal A of R such that $v(A) = 0$ for all $v \in F$, and (iv) holds.

We note that condition (iv) insures that proper prime ideals are maximal and that valuation rings R_v are identical with the quotient rings R_P where P runs over all minimal primes of R. We call F the family of essential valuations of R. In the remainder of this section, R denotes an AD-domain, F the family of essential valuations of R.

We state the following lemmas:

Lemma 3. 3. *If $A, B \in I(R)$, $v \in F$, then $v(AB) = v(A) + v(B)$.*

Lemma 3. 4. *For any principal fractionary ideal Rx and any $v \in F$, $v(Rx) = v(x)$.*

Now, for $A, B \in I(R)$, define $A \sim B$ iff $v(A) = v(B)$ for all $v \in F$. Then $\sim$ is an equivalence relation on $I(R)$. For $A \in I(R)$ we let $[A]$ denote the equivalence class of A with respect to $\sim$ and we let $\mathscr{A}(R)$ denote the set of all such equivalence classes. We define $+$ on $\mathscr{A}(R)$ by $[A]+[B] = [AB]$, and we define $\leqq$ on $\mathscr{A}(R)$ by $[A] \leqq [B]$ iff $v(A) \leqq v(B)$ for each $v \in F$. With these definitions $\mathscr{A}(R)$ is a commutative, partially ordered semigroup with identity $0 = [R]$.

Lemma 3. 5. *Let $A \in I(R)$. Then, considering $[A]$ and $\operatorname{div}_R(A)$ as subsets of $I(R)$, $[A] \subseteqq \operatorname{div}_R(A)$.*

Proof. Let $B \in [A]$. Then $v(B) = v(A)$ for each $v \in F$. If $A \subseteqq Rx$, then $v(B) = v(A) \geqq v(Rx) = v(x)$ for each $v \in F$. Thus if $b \in B$, $v(b) - v(x) \geqq 0$ for all $v \in F$; i.e., $v\left(\frac{b}{x}\right) \geqq 0$ for all $v \in F$, so $\frac{b}{x} \in \bigcap_{v \in F} R_v = R$, and hence $B \subseteqq Rx$. Similarly, if $B \subseteqq Ry$ then $A \subseteqq Ry$. Then $\tilde{A} = \tilde{B}$ and $\operatorname{div}_R(A) = \operatorname{div}_R(B)$, But then $B \in \operatorname{div}_R(A)$.

Proposition 3. 6. *The map $g: \mathscr{A}(R) \to \mathscr{D}(R)$, defined by $g([A]) = \operatorname{div}_R(A)$, is an order-preserving homomorphism of the partially ordered semigroup $\mathscr{A}(R)$ onto the lattice-ordered group $\mathscr{D}(R)$.*

Proof. Lemma 3. 5 shows that g is well-defined and onto. It is easy to check that g is a homomorphism. To see that g preserves order, recall that $\operatorname{div}_R(A) \leqq \operatorname{div}_R(B)$ iff $A \prec B$. We shall show that if $[A] \leqq [B]$, then $A \prec B$. Thus suppose $[A] \leqq [B]$ and let $A \subseteqq Rx$. Then $v(B) \geqq v(A) \geqq v(x)$ for each $v \in F$ since $[A] \leqq [B]$. As in the proof of 3. 5 we have $B \subseteqq Rx$ so that $A \prec B$.

We can now prove the following theorem.

Theorem 3. 7. *Let R be an AD-domain with family F of essential valuations. The following statements are equivalent.*

(1) *Every proper prime ideal of R is divisorial.*

(2) *Every fractionary ideal of R is divisorial.*

(3) *R is a Dedekind domain.*

(4) *$\mathscr{A}(R)$ is a group.*

(5) *The map g of $\mathscr{A}(R)$ onto $\mathscr{D}(R)$ is an isomorphism.*

Proof. $(1) \Leftrightarrow (3)$. This is corollary 2. 2.

$(2) \Rightarrow (5)$ If every fractionary ideal is divisorial, then for $A \in I(R)$ we have $\operatorname{div}_R(A) = \{A\}$. But $A \in [A] \subseteqq \operatorname{div}_R(A) = \{A\}$. It follows that, considered as sets, $[A] = \operatorname{div}_R(A)$ so that g is 1—1, hence an isomorphism.

(5)$\Rightarrow$(4) Clear since $\mathscr{D}(R)$ is a group.

(4)$\Rightarrow$(3) Suppose $\mathscr{A}(R)$ is a group. Then if $[A]\in\mathscr{A}(R)$, there is $[B]\in\mathscr{A}(R)$ such that $[A]+[B] = 0$; i.e., $[AB]=[R]$. Then $v(AB)=0$ for all $v\in F$ and so by theorem 3. 2, $AB=R$. Thus A is invertible and hence has a finite basis so that R is Noetherian and therefore Dedekind.

(3)$\Rightarrow$(2) This is found in [1], page 23.

4. Some further examples of AD-domains

Let R be an AD-domain with quotient field K and family F of essential valuations. Let L be a subfield of K and put $A = R\cap L$. We may suppose that L is the quotient field of A, for if T is the quotient field of A then $A\subseteqq T\subseteqq L$, and so $A\subseteqq R\cap T\subseteqq R\cap L = A$.

Proposition 4. 1. *If R is integral over A, then A is almost-Dedekind.*

Proof. For $v\in F$, let v' denote the restriction of v to L. Let $F'=\{v'|v\in F$ and v is nontrivial on $L\}$. Thus if $v'\in F'$, then $v'(x)\neq 0$ for some non-zero element $x\in L$. It is clear that each $v'\in F'$ is a rank one discrete valuation since each $v\in F$ is a rank one discrete valuation. Since $R=\bigcap_{v\in F} R_v$ we have $A=R\cap L=\big(\bigcap_{v\in F} R_v\big)\cap L=\bigcap_{v\in F}(R_v\cap L)=$ $=\bigcap_{v'\in F}(R_v\cap L)$, for if $w\in F$ is trivial on L, then $L\subseteqq R_w$. For $v'\in F'$, let $A_{v'}=$ $=\{x\in L|v'(x)\geqq 0\}$, and let $Q(v')=\{y\in A|v'(y)>0\}$. Then $Q(v')=A\cap P(v)$, so $Q(v')$ is a non-zero minimal prime of A since $P(v)$ is a non-zero minimal prime of R and R is integral over A. We have $A_{Q(v')}=L\cap R_{P(v)}=L\cap R_v=A_{v'}$ for each $v'\in F'$. For if $\frac{x}{y}\in A_{Q(v)}$, then $v'\left(\frac{x}{y}\right)\geqq 0$; i.e., $\frac{x}{y}\in L\cap R_v=L\cap R_{P(v)}=A_{v'}$. Thus $A_{Q(v')}\subseteqq A_{v'}$. On the other hand, let $\frac{x}{y}\in A_{v'}$, $x, y\in A$. Then $v'\left(\frac{x}{y}\right)\geqq 0$. Since v' is a non-trivial, rank one discrete valuation on L, there is an irreducible element $\pi\in A_{v'}$, such that $\frac{x}{y}=\frac{\pi^n u}{w}$, where $n\geqq 0$ is an integer, $u, w\in A$ are such that $v(u)=0=v(w)$ and $\pi\in A$ without loss of generality. Then $w\in Q(v')$, and so $\frac{x}{y}\in A_{Q(v)}$; i.e., $A_{v'}=A_{Q(v')}$. Now let B be an ideal of A such that $v'(B)=0$ for all $v'\in F'$, where $v'(B)=\inf_{b\in B} v'(b)$, for each $v'\in F'$. Then $v(BR)=0$ for all $v\in F$. For if $v\in F$ is trivial on L, then $v(BR)=0$ clearly. If $v\in F$ is such that $v'\in F'$, then $v'(B)=v(BR)=0$. Thus BR is an ideal of R such that $v(BR)=0$ for all $v\in F$, and so $BR=R$. In this case, $B=A$. For if $B<A$, then $B\subseteqq M'$ for some maximal ideal M' of A. Since R is integral over A, there is a maximal ideal M' of R such that $M\cap A = M'$. Then $B\subseteqq M$, and so $BR\subseteqq M<R$, a contradiction. So we must have $B=A$. Thus F' is a family of valuations on L satisfying the conditions of theorem 3. 2.

Proposition 4. 2. *Let R be an AD-domain and let x be an indeterminate. Let S denote the multiplicatively closed set of $R[x]$ consisting of all monic polynomials. Then $(R[x])_S$ is AD.*

PROOF. Denote $(R[x])_S$ by R^1. *Let $\mathscr{P}$ be a prime ideal in R^1, $(0) < \mathscr{P} < R^1$, and consider $R^1_{\mathscr{P}}$.*

Case 1. $\mathscr{P} \cap R = (0)$. Then $K[x] \subseteqq R^1_{\mathscr{P}}$ and $K[x]$ is Dedekind. Hence $R^1_{\mathscr{P}}$ is Dedekind.

Case 2. $\mathscr{P} \cap R = P \neq (0)$. Then P is a prime ideal in R, and $\mathscr{P} \cap R[x] = PR[x]$. For if not, then $\mathscr{P} \cap S \neq 0$, and $\mathscr{P} = R^1$, a contradiction.

Let $M = R^1 - \mathscr{P}$. Then $M_1 = R[x] - PR[x] \subseteqq M$. Now $R[x]_P = R_P[x] \subseteqq R^1_{\mathscr{P}}$. Let S' denote the set of monic polynomials in $R_P[x]$. Then $(R_P[x])_{S'}$ is Dedekind since R_P is Dedekind by Lemma 2. 1 of [2]. We have $(R_P[x])_{S'} \subseteqq R^1_{\mathscr{P}}$. For, let $z \in (R_P[x])_{S'} = (R[x]_P)_{S'}$. Then

$$z = \frac{f(x)}{m} \Big/ x^n + \frac{a_{n-1}}{m_{n-1}} x^{n-1} + \dots + \frac{a_0}{m_0}, \quad \text{where } f(x) \in R[x], m, m_0, \dots, m_{n-1} \in R - P,$$
$$a_0, \dots, a_{n-1} \in R.$$

Then $z = \dfrac{f(x)}{m} \Big/ [(m'x^n + a'_{n-1}x^{n-1} + \dots + a'_0)/m']$, where $m' = m_0 m_1 \dots m_{n-1}$. Then $z = m'f(x)/(mm'x^n + \dots + ma'_0)$ and $g(x) = mm'x^n + \dots + ma'_0$ does not belong to $\mathscr{P}$ since $mm' \in R - P$, and so $g(x) \in R[x] - PR[x]$. Thus, $z \in R^1_{\mathscr{P}}$. Since $(R_P[x])_{S'} \subseteqq R^1_{\mathscr{P}}$ and $(R_P[x])_{S'}$ is Dedekind, $R^1_{\mathscr{P}}$ is also Dedekind.

Author's Note: This paper constitutes part of a Ph. D. dissertation written under the direction of Professor PAUL J. MCCARTHY at the University of Kansas. The author wishes to express his appreciation to Professor McCarthy for his counsel and advice during the course of this work.

References

[1] N. BOURBAKI, Algebre Commutative, *Paris*, 1965.
[2] LUTHER CLABORN, Dedekind domains and rings of quotients, *Pacific J. Math.*, **15** (1965), 59—64.
[3] R. W. GILMER, JR., Integral domains which are almost Dedekind, *Proc. Amer. Math. Soc.*, **15** (1964), 813—818.
[4] O. ZARISKI and P. SAMUEL, Commutative Algebra, Vol. I, *Princeton, New Jersey*, 1958.
[5] ———, Commutative Algebra, Vol. II, *Princeton, New Jersey*, 1960.
[6] M. NAGATA, Local rings, *New York*, 1962.
[7] R. W. GILMER, Multiplicative ideal theory, *Kingston*, 1968.

(Received Oktober 31, 1968.)