

On various Boolean structures in a given Boolean algebra

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HANNA NEUMANN proved in [1] the conjecture of A. KERTÉSZ concerning various group operations in a given group. In this note we solve a similar problem for Boolean algebra.

1. Consider a Boolean algebra $\mathfrak{B} = (B, \vee, \wedge, ')$ with unity element I and zero element 0 , join $x \vee y$, meet $x \wedge y$, and complement x' . Let a be an arbitrary element of B . Define two binary operations $x \cup y$ and $x \cap y$ by formulas

$$(1) \quad x \cup y = [a \wedge (x \vee y)] \vee [a' \wedge x \wedge y]$$

and

$$(2) \quad x \cap y = (x' \cup y')' = [a' \vee (x \wedge y)] \wedge [a \vee x \vee y].$$

Theorem 1. *The algebra $\mathfrak{B}^a = (B, \cup, \cap, ')$ is a Boolean algebra with unity element a , zero a' , join $x \cup y$, meet $x \cap y$ and complement x' . The mapping $\Phi: B \rightarrow B$ defined by formula*

$$(3) \quad \Phi(x) = (a \wedge x) \vee (a' \wedge x') = (I \cap x) \cup (0 \cap x')$$

is an involutory isomorphism $\Phi: \mathfrak{B} \rightarrow \mathfrak{B}^a$.

PROOF. We shall prove first that Φ is an involutory mapping, and therefore one-to-one and onto. Indeed, we have

$$\begin{aligned} \Phi[\Phi(x)] &= [a \wedge \Phi(x)] \vee [a' \wedge \Phi(x)'] = \\ &= (a \wedge [(a \wedge x) \vee (a' \wedge x')]) \vee (a' \wedge [(a \wedge x) \vee (a' \wedge x')]) = \\ &= (a \wedge x) \vee [a' \wedge (a' \vee x) \wedge (a \vee x)] = (a \wedge x) \vee (a' \wedge x) = x. \end{aligned}$$

Then we prove that

$$(4) \quad \Phi(x)' = \Phi(x')$$

and

$$(5) \quad \Phi(x) \cup \Phi(y) = \Phi(x \vee y).$$

Indeed

$$\Phi(x)' = [(a \wedge x) \vee (a' \wedge x')] = (a' \vee x') \wedge (a \vee x) = (a \wedge x') \vee (a' \wedge x) = \Phi(x').$$

Further

$$\begin{aligned}\Phi(x) \cup \Phi(y) &= (a \wedge [\Phi(x) \vee \Phi(y)]) \vee [a' \wedge \Phi(x) \wedge \Phi(y)] = \\ &= (a \wedge [(a \wedge x) \vee (a' \wedge x') \vee (a \wedge y) \wedge (a' \wedge y')]) \vee \\ &\vee (a' \wedge [(a \wedge x) \vee (a' \wedge x')] \wedge [(a \wedge y) \vee (a' \wedge y')]) = \\ &= [a \wedge (x \vee y)] \vee (a' \wedge x' \wedge y') = \Phi(x \vee y).\end{aligned}$$

Therefore, Φ is an isomorphism of the algebra $(B; \vee, ')$ onto $(B, \cup, ')$. This isomorphism sends the unit I into $(a \wedge I) \vee (a' \wedge I') = a$, and consequently $0 \rightarrow a'$. Due to (2), we have also $\Phi(x \wedge y) = \Phi(x) \cap \Phi(y)$.

By a relatively simple computation one checks that $\Phi(x) = (I \cap x) \cup (0 \cap x')$. Indeed,

$$\begin{aligned}I \cap x &= [a' \vee x] \wedge [a \vee I \vee x] = a' \vee x, \\ 0 \cap x' &= [a' \vee 0] \wedge (a \vee x') = a' \wedge x'.\end{aligned}$$

Now

$$\begin{aligned}(I \cap x) \cup (0 \cap x') &= (a' \vee x) \cup (a' \wedge x') = \\ &= (a \wedge [a' \vee x \vee (a' \wedge x')]) \vee [a' \wedge (a' \vee x) \wedge a' \wedge x'] = (a \wedge x) \vee (a' \wedge x') = \Phi(x).\end{aligned}$$

Corollary. *The join and meet of the algebra $\mathfrak{B}$ can be expressed by the operations of the algebra $\mathfrak{B}^a$ by formulae*

$$\begin{aligned}(1') \quad x \vee y &= [I \cap (a \cup y)] \cup [0 \cap x \cap y] \\ (2') \quad x \wedge y &= (x' \vee y')' = [0 \cup (a \cap y)] \cap [I \cup x \cup y]\end{aligned}$$

PROOF. Applying theorem 1 with I taking the place of a to the algebra $\mathfrak{B}^a$ we obtain another algebra with join and meet

$$\begin{aligned}x \cup y &= [I \cap (x \cup y)] \cup [0 \cap x \cap y], \\ x \cap y &= [I \cup x \cup y] \cap [0 \cup (x \cap y)],\end{aligned}$$

which is an isomorphic image of the algebra $\mathfrak{B}^a$ under the isomorphism $x \rightarrow (a \cap x) \cup (a' \cup x')$. But this isomorphism is nothing else than $\Phi = \Phi^{-1}$. Therefore, the isomorphic image of $\mathfrak{B}^a$ coincides with $\mathfrak{B}$ and $x \cup y = x \vee y$, $x \cap y = x \wedge y$, and so formulae (1'), (2') hold.

Theorem 2. *The binary operations $\vee$, $\wedge$, $\cup$, $\cap$ are distributive with respect to each other.*

The proof is a matter of simple computations.

2. The question arises whether or not the Boolean algebras $\mathfrak{B}^a$ are the only Boolean algebras on the set B whose operations can be expressed in terms of the operations of $\mathfrak{B}$ and constants. We shall give an affirmative answer to this question.

We start with the following lemmas.

Lemma 1. *Any unary operation $\bar{x}$ in B which can be expressed in terms of the original operations of $\mathfrak{B}$ and constants, and which satisfies the identity $\bar{\bar{x}} = x$ must be of the form*

$$(6) \quad \bar{x} = (v' \wedge x) \vee (v \wedge x'),$$

where $v \in B$ is some constant.

PROOF. The most general word in one variable with constants is

$$\bar{x} = (u \wedge x) \vee (v \wedge x') \vee c$$

(u, v, c — constants). The condition $\bar{x} = y$ yields

$$\begin{aligned} x = \bar{x} &= (u \wedge [(u \wedge x) \vee (v \vee x') \vee c]) \vee [v \wedge (u' \vee x') \wedge (v' \vee x) \wedge c'] \vee c = \\ &= (u \wedge x) \vee (u \wedge v \wedge x') \vee (u \wedge c) \vee (v \wedge u' \wedge x \wedge c') \vee c = \\ &= ([u \vee (v \wedge c')] \wedge x) \vee (u \wedge v \wedge x') \vee c. \end{aligned}$$

Setting $x=0$ we obtain $0 = (u \wedge v) \vee c$, which implies $c=0$ and $u \wedge v = 0$. Setting $x=I$, $c=0$, we obtain $u \vee v = I$. Consequently $u=v'$ which proves (6).

Lemma 2. Any binary operation $x \cup y$ in B , which can be expressed in terms of x, y , the operations of $\mathfrak{B}$ and constants, and which satisfies the conditions

$$(7) \quad 0 \cup 0 = 0, \quad 0 \cup I = I, \quad I \cup 0 = I \quad \text{and} \quad I \cup I = I,$$

coincides with the operation $x \vee y$.

PROOF. The most general binary operation which can be so expressed is

$$\begin{aligned} x \cup y &= a \vee (b \wedge x) \vee (c \wedge y) \vee (d \wedge x') \vee (e \wedge y') \vee (f \wedge x \wedge y) \vee \\ &\vee (g \wedge x' \wedge y) \vee (h \wedge x \wedge y') \vee (j \wedge x' \wedge y') \end{aligned}$$

with constants $a, b, c, d, e, f, g, h, j$.

The condition $0 \cup 0 = 0$ implies that

$$a=d=e=j=0,$$

whence

$$x \cup y = (b \wedge x) \vee (x \wedge y) \vee (f \wedge x \wedge y) \vee (g \wedge x \wedge y') \vee (h \wedge x' \wedge y).$$

One can easily check by computations that

$$\begin{aligned} (b \wedge x) \vee (c \wedge y) \vee (f \wedge x \wedge y) &= (b \wedge x) \vee (c \wedge y) \vee (f \wedge b' \wedge c' \wedge x \wedge y), \\ (b \wedge x) \vee (g \wedge x \wedge y') &= (b \wedge x) \vee (g \wedge b' \wedge x \wedge y'), \\ (c \wedge x) \vee (h \wedge x' \wedge y) &= (c \wedge x) \vee (h \wedge c' \wedge x' \wedge y). \end{aligned}$$

Therefore the constant f can be replaced by $f \wedge b' \wedge c'$ disjoint with $b \vee c$, g can be replaced by $g \wedge b'$ disjoint with b , and h can be replaced by $h \wedge c'$ disjoint with c . Thus we can assume that

$$(8) \quad f \wedge (b \vee c) = 0, \quad g \wedge b = 0, \quad h \wedge c = 0$$

without loss of generality.

The conditions $I \cup 0 = I$, $0 \cup I = I$ and $I \cup I = I$ imply

$$b \vee g = I, \quad c \vee h = I, \quad b \vee c \vee f = I$$

which together with (8) yields

$$g=b', \quad h=c', \quad f= b' \wedge c'.$$

Consequently we have

$$x \cup y = (b \wedge x) \vee (c \wedge y) \vee (b' \wedge c' \wedge x \wedge y) \vee (b' \wedge x \wedge y') \vee (c' \wedge x' \wedge y).$$

But

$$\begin{aligned} (b' \wedge c' \wedge x \wedge y) \vee (b' \wedge x \wedge y') &= b' \wedge x \wedge [(c' \wedge y) \vee y'] = \\ &= (b' \wedge x) \wedge (c' \vee y') = (b' \wedge x) \wedge (c \wedge y)', \end{aligned}$$

and

$$(b' \wedge c' \wedge x \wedge y) \vee (c' \wedge x' \wedge y) = (c' \wedge y) \wedge (b \wedge x)',$$

Further

$$(b \wedge x) \vee [(x' \wedge y) \wedge (b \wedge x)'] = (b \wedge x) \vee (c' \wedge y),$$

$$(c \wedge y) \vee [(b' \wedge x) \wedge (c \wedge y)'] = (c \wedge y) \vee (b' \wedge x).$$

Consequently,

$$x \cup y = (b \wedge x) \vee (c' \wedge y) \vee (c \wedge y) \vee (b' \wedge x) = x \vee y,$$

Theorem 3. *Any two Boolean structures on B , with coinciding zeros, and such that the Boolean operations of one of them can be expressed in terms of the Boolean operations of the other and constants, coincide.*

PROOF. We can assume without loss of generality that one of the Boolean structures is the original algebra $\mathfrak{B}$. Let the other algebra be $\bar{\mathfrak{B}} = (B, \cup, \cap, -)$ with join $x \cup y$, meet $x \cap y$ and complement $\bar{x}$. Since 0 is assumed to be the zero of $\bar{\mathfrak{B}}$, the identities (7) will be satisfied, and therefore, by lemma 2,

$$x \cup y = x \vee y.$$

Now, by lemma 1,

$$\bar{x} = (v' \wedge x) \vee (v \wedge x').$$

Since 0 is the zero of the algebra $\bar{\mathfrak{B}}$, its unity will be $\bar{0} = (v' \wedge 0) \vee (v \wedge 0') = v$. Consequently v must satisfy the condition $v \cup x = v$ for every x in particular

$$v = v \cup v' = v \vee v' = I.$$

Thus

$$\bar{x} = (I' \wedge x) \vee (I \wedge x') = x'.$$

This proves the theorem.

Now we are ready to prove the main result of the paper:

Theorem 4. *Every Boolean algebra $\mathfrak{B}^*$ with the underlying set B whose operations can be expressed in terms of the operations of the Boolean algebra $\mathfrak{B}$ and constants coincide with one of the algebras $\mathfrak{B}^a$ of theorem 1.*

PROOF. Consider the algebra $\mathfrak{B}^*$ and let a' be its zero element. By theorem 1, the algebra $\mathfrak{B}^a$ is then another Boolean algebra on B with the same zero element a' . Since the operations of $\mathfrak{B}$ can be expressed in terms of the operations of $\mathfrak{B}^a$ by the corollary to theorem 1, also the operations of $\mathfrak{B}^*$ are expressible in terms of the operations of $\mathfrak{B}^a$ and constants. By theorem 3 the two algebras $\mathfrak{B}^a$ and $\mathfrak{B}^*$ with coinciding zero element a' must coincide. This proves the theorem.

Remark. Theorem 3 is a strengthening of Theorem 1 of T. TRACZYK [2] in which the assertion of theorem 3 is obtained under the additional assumptions that the expressions for the operations do not involve constants other than I and 0 , and that I is the unity of both algebras.

References

- [1] HANNA NEUMANN, On a question of Kertész, *Publ. Math. Debrecen* **8** (1961), 75—78.
- [2] T. TRACZYK, Weak isomorphisms of Boolean and Post algebras, *Colloquium Math.* **13** (1965), (1965), 159—164.

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