

A theorem on bounded analytic functions

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Introduction. Let $f(z) = a_1 z + a_2 z^2 + \dots$, $0 < a_1 \leq 1$, be analytic in the unit disc $E = \{z: |z| < 1\}$ and satisfy $|f(z)| \leq 1$ in E . Since $\operatorname{Re} f'(0) = a_1 > 0$, it follows that there exists a positive number r_0 such that $\operatorname{Re} f'(z) > 0$ whenever $|z| < r_0$. The purpose of this note is to determine the number r_0 .

Theorem. Let $f(z) = a_1 z + a_2 z^2 + \dots$, $0 < a_1 \leq 1$, be analytic in E and satisfy $|f(z)| \leq 1$ for all z in E . Then $\operatorname{Re} f'(z) > 0$ in $|z| < r_0$, where

$$r_0 = \frac{a_1}{1 + \sqrt{1 - a_1^2}}.$$

The number r_0 cannot be replaced by a larger one.

PROOF. Define

$$F(z) = \frac{\frac{f(z)}{z} - a_1}{1 - a_1 \frac{f(z)}{z}}.$$

Then $F(z)$ is analytic in E , $F(0) = 0$ and $|F(z)| \leq 1$ in E . Therefore, by Schwarz' Lemma it follows that

$$(1) \quad \left| \frac{\frac{f(z)}{z} - a_1}{1 - a_1 \frac{f(z)}{z}} \right| \leq |z|$$

for all $z \in E$.

Squaring both the sides of (1) we obtain

$$\begin{aligned} \left| \frac{f(z)}{z} - a_1 \right|^2 &= \left| \frac{f(z)}{z} \right|^2 - 2a_1 \operatorname{Re} \frac{f(z)}{z} + a_1^2 \leq \\ &\leq r^2 \left[1 - 2a_1 \operatorname{Re} \frac{f(z)}{z} + a_1^2 \left| \frac{f(z)}{z} \right|^2 \right], \quad |z| = r. \end{aligned}$$

Or

$$(1 - a_1^2 r^2) \left| \frac{f(z)}{z} \right|^2 - (1 - r^2) 2a_1 \operatorname{Re} \frac{f(z)}{z} \leq r^2 - a_1^2.$$

Or

$$\left| \frac{f(z)}{z} \right|^2 - \frac{(1 - r^2)}{1 - a_1^2 r^2} \cdot 2a_1 \operatorname{Re} \frac{f(z)}{z} + \frac{(1 - r^2)^2 a_1^2}{(1 - a_1^2 r^2)} \leq \frac{r^2 - a_1^2}{1 - a_1^2 r^2} + \frac{(1 - r^2)^2 a_1^2}{(1 - a_1^2 r^2)^2}.$$

I.e.

$$\left| \frac{f(z)}{z} - \frac{(1 - r^2)}{1 - a_1^2 r^2} a_1 \right|^2 \leq \frac{r^2 (1 - a_1^2)^2}{(1 - a_1^2 r^2)^2},$$

or

$$(2) \quad \left| \frac{f(z)}{z} - \frac{(1 - r^2) a_1}{1 - a_1^2 r^2} \right| \leq \frac{r(1 - a_1^2)}{(1 - a_1^2 r^2)}.$$

Put

$$(3) \quad G(z) = \frac{f(z)}{z}.$$

Then $G(z)$ is analytic in E . Also, since $|f(z)| \leq |z|$ ($f(z)$ satisfies the hypotheses of Schwarz's Lemma) in E , it follows that $|G(z)| \leq 1$ in E . Also we have

$$(4) \quad |G'(z)| \leq \frac{1 - |G(z)|^2}{1 - r^2}$$

([1], p. 168, equ. (28)) for all z in E .

Differentiating (3) and making use of (4) we obtain the following inequality:

$$(5) \quad \left| f'(z) - \frac{f(z)}{z} \right| \leq \frac{r \left[1 - \left| \frac{f(z)}{z} \right|^2 \right]}{1 - r^2}.$$

From (5) we see that $f'(z)$ lies in the circle whose centre is at $f(z)/z$ and whose radius is equal to $r \{1 - |f(z)/z|^2\}/(1 - r^2)$. Therefore, we conclude that

$$\begin{aligned} \operatorname{Re} f'(z) &\leq \operatorname{Re} \frac{f(z)}{z} - r \frac{\left[1 - \left| \frac{f(z)}{z} \right|^2 \right]}{1 - r^2} = \frac{r}{(1 - r^2)} \left[\left| \frac{f(z)}{z} \right|^2 + \frac{1 - r^2}{r} \operatorname{Re} \frac{f(z)}{z} - 1 \right] = \\ (6) \quad &= \frac{r}{(1 - r^2)} \left[\left| \frac{f(z)}{z} \right|^2 + \frac{1 - r^2}{r} \operatorname{Re} \frac{f(z)}{z} + \frac{(1 - r^2)^2}{4r^2} - 1 - \frac{(1 - r^2)^2}{4r^2} \right] = \\ &= \frac{r}{(1 - r^2)} \left[\left| \frac{f(z)}{z} + \frac{1 - r^2}{2r} \right|^2 - \frac{(1 + r^2)}{4r^2} \right]. \end{aligned}$$

Now, from (2) we have

$$\left| \left\{ \frac{f(z)}{z} + \frac{1 - r^2}{2r} \right\} - \left\{ \frac{a_1(1 - r^2)}{1 - a_1^2 r^2} + \frac{1 - r^2}{2r} \right\} \right| \leq \frac{r(1 - a_1^2)^2}{1 - a_1^2 r^2},$$

from which it follows that

(7)

$$\left| \frac{f(z)}{z} + \frac{1-r^2}{2r} \right|^2 \cong \left[\frac{a_1(1-r^2)}{1-a_1^2 r^2} + \frac{1-r^2}{2r} - \frac{r(1-a_1^2)}{1-a_1^2 r^2} \right]^2 = \frac{(1+a_1 r-3r^2+a_1 r^3)^2}{4r^2(1-a_1 r)^2}.$$

Making use of (7) in (6) we conclude that $\operatorname{Re} f'(z) > 0$ provided

$$\frac{r}{(1-r^2)} \left[\frac{(1+a_1 r-3r^2+a_1 r^3)^2}{4r^2(1-a_1 r)^2} - \frac{(1+r^2)^2}{4r^2} \right] > 0,$$

or

$$\frac{a_1 r^2 - 2r + a_1}{(1-a_1 r)^2} > 0,$$

i.e.

$$r < \frac{a_1}{1 + \sqrt{1-a_1^2}} = r_0.$$

To show that our result is sharp, we consider the function

$$f_0(z) = \frac{z(a_1 - z)}{1 - a_1 z}, \quad 0 < a \leq 1,$$

which satisfies the hypotheses of our theorem. A little computation reveals that $f'_0(z) = 0$ when $z = r_0$. This shows that the number r_0 cannot be replaced by any larger one. The theorem is therefore proved.

Now it is known ([2], [3]) that if $F(z)$ is regular in a convex domain D and if $\operatorname{Re} F'(z) > 0$ in D then $F(z)$ is univalent in D . We, therefore, have the following corollary:

Corollary: If $f(z)$ satisfies the hypotheses of our theorem then $f(z)$ is univalent in $|z| < r_0$.

References

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