On the generalized uniform distribution (mod 1)

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Suppose that $0 \leqq s_n \leqq 1$ for every n, and denote the interval $0 \leqq a \leqq x \leqq b \leqq 1$ by I. We denote by $I(x)$ the characteristic function of I which is 1 in I and 0 elsewhere.

A sequence $\{s_k\}$ is then said to be uniformly distributed if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} I(s_k) = b-a \tag{1}$$

for every I,

By a well known theorem of WEYL [1], the condition (1) may be expressed alternatively as

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} e(hs_k) = 0, \qquad h=1, 2, \dots, \tag{2}$$

where $e(t)$ denotes $e^{2\pi it}$.

In other words, the sequence $\{s_k\}$ is uniformly distributed if and only if the sequence $\{e(hs_k)\}$ is $(C, 1)$ summable to the value zero for every $h=1, 2, \dots$.

E. HLAWKA [2] has introduced the general concept of A-uniform distribution, where A is an arbitrary regular summability method. If $A \equiv (a_{nk})$ then the sequence $\{s_k\}$ is said to be A-uniformly distributed if and only if the sequence $\{e(hs_k)\}$ is A-summable to zero for every $h=1, 2, \dots$.

J. CIGLER [3] considered the problem of determining the summation methods with respect to which the sequences $\{n\theta\}$ are uniformly distributed mod 1 for each irrational number θ, where $\{n\theta\} = n\theta - [n\theta]$.

Here we discuss the same problem for the sequences $\{n^2\theta\}$. To do this we require the following results of J. H. B. KEMPERMAN [4].

Let $A \equiv (a_{nk})$ denote a fixed nonnegative regular summation method; $n, k = 1, 2, \dots$. For each sequence $Z = (Z(k))$ of complex numbers with

$$\|Z\| = \sup_k |Z(k)| < \infty,$$

put

$$\alpha_n\{Z(k)\} = \sum_{k=1}^{\infty} a_{nk} Z(k).$$

Let

$$\|\alpha_n\| = \sum_{k=1}^{\infty} a_{nk} \quad \text{and} \quad \|\beta_n\| = \sum_{k=1}^{\infty} |a_{n,k} - a_{n,k+1}|.$$

Lemma 1. *Suppose that $\|Z\| \leqq 1$. Then, provided $\|\alpha_n\| > 0$, we have for each positive integer m that*

$$\|\alpha_n\|^{-1} |\alpha_n\{Z(k)\}|^2 \leqq \frac{4}{3} m \|\beta_n\| + \frac{1}{m} \|\alpha_n\| + \frac{2}{m} \sum_{h=1}^{m-1} \left(1 - \frac{h}{m}\right) \operatorname{Re}[\alpha_n\{Z(k+h)\overline{Z(k)}\}] \tag{3}$$

In the particular case where $\lim_{n \to \infty} \|\beta_n\| = 0$, we have:

Lemma 2. *Suppose that $\|Z\| < \infty$. Let*

$$\varliminf_{m \to \infty} \varlimsup_{n \to \infty} \frac{2}{m} \sum_{h=1}^{m-1} \left(1 - \frac{h}{m}\right) \operatorname{Re}[\alpha_n\{Z(k+h)\overline{Z(k)}\}] = 0. \tag{4}$$

Then

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} Z(k) = 0 \tag{5}$$

Using Hölder's inequality, it follows that a sufficient condition for (4) to hold is that for some $1 \leqq r < \infty$,

$$\varliminf_{m \to \infty} \varlimsup_{n \to \infty} \frac{1}{m} \sum_{h=1}^{m-1} |\alpha_n\{Z(k+h)\overline{Z(k)}\}|^r = 0 \tag{6}$$

Let $p_0 > 0$, $p_n \geqq 0$, and let $P_n = p_0 + p_1 + p_2 + \cdots + p_n$. Let $\frac{p_n}{P_n} \to 0$ as $n \to \infty$. A sequence s_k is (N, p_n) summable to s if

$$\frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \to s$$

when $n \to \infty$.

We now prove the following result:

Theorem 1. *Let p_n satisfy, besides the above properties, the conditions:*
(i) *p_n decreases as n increases and $P_n \to \infty$,*
(ii) $\frac{1}{P_m^r} \sum_{n=0}^{m} p_n^r \to 0$ as $m \to \infty$
for some $1 \leqq r < \infty$.
Then the sequence $\{n^2 \theta\}$ is (N, p_n) uniformly distributed.

Proof. Taking $a_{m,n} = \frac{p_{m-n}}{P_m}$, $n \leqq m$, $a_{m,n} = 0$ for $n > m$, we first show that

$$\lim_{m \to \infty} \sum_{n=0}^{\infty} |a_{m,n} - a_{m,n+1}| = 0.$$

Here we have:

$$\sum_{n=0}^{\infty} |a_{m,n} - a_{m,n+1}| = \frac{1}{P_m} \sum_{n=0}^{m-1} |p_{m-n} - p_{m-n-1}| + \frac{|p_0|}{P_m} =$$

$$= \frac{1}{P_m} [p_0 - p_m] + \frac{p_0}{P_m} \to 0 \quad \text{as} \quad m \to \infty.$$

Secondly we prove that

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{p_{n-k}}{P_n}e^{2\pi ik^2\theta}=0.$$

We take $Z(k)=e^{2\pi ik^2\theta}$.

$$Z(k+h)\overline{Z(k)}=e^{2\pi i[(k+h)^2-k^2]\theta}.$$

So we have

$$|\alpha_n\{Z(k+h)\overline{Z(k)}\}|^r=\frac{1}{P_n^r}\left|\sum_{k=0}^{n}p_{n-k}e^{2\pi i[(k+h)^2-k^2]\theta}\right|^r\leqq$$

$$\leqq\frac{1}{P_n^r}\sum_{k=0}^{n}p_{n-k}^r=\frac{1}{P_n^r}\sum_{k=0}^{n}p_k^r\to 0$$

as $n\to\infty$ for some $1\leqq r<\infty$.

It follows that

$$\frac{1}{m}\sum_{h=1}^{m-1}|\alpha_n\{Z(k+h)\overline{Z(k)}\}|^r\leqq\frac{1}{m}\sum_{h=1}^{m-1}\left(\sum_{k=0}^{n}\frac{p_k^r}{P_n^r}\right)\to 0$$

as $m\to\infty$. Hence condition (6) is satisfied provided that conditions (i) and (ii) are satisfied. It follows from Lemma (2) that

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{p_{n-k}}{P_n}e^{2\pi ik^2\theta}=0$$

and the sequence $\{n^2\theta\}$ is (N,p_n) uniformly distributed mod 1.

As an example of a method (N,p_n) satisfying the above conditions is that one defined by the sequence $p_n=\dfrac{1}{n+1}$. Here we can take r any number greater than 1. It is well known that this method is weaker than any Cesaro mean of positive order, [5].

The regular Riesz weighted means (R,p_k) are defined by

$$\lim_{n\to\infty}\frac{p_0s_0+p_1s_1+\cdots+p_ns_n}{P_n},$$

where $P_n=p_0+p_1+\cdots+p_n$.

We now prove the following result:

Theorem 2. *If (R,p_n) is a regular Riesz mean which satisfies the additional conditions:*

$$p_{n+1}\geqq p_n>0\quad\textit{for all}\quad n=0,1,2,\ldots,\tag{7}$$

$$\lim_{n\to\infty}\frac{p_n}{P_n}=0,\tag{8}$$

$$\lim_{n\to\infty}\frac{1}{P_n^r}\sum_{k=0}^{n}p_k^r=0\quad\textit{for some}\quad 1\leqq r<\infty,\tag{9}$$

then the sequence $\{n^2\theta\}$ is (R,p_n) uniformly distributed (mod 1).

PROOF. Taking $a_{m,n} = \frac{p_n}{P_m}$ for $n \leqq m$, $a_{m,n} = 0$ for $n > m$ we get, as before, from conditions (7) and (8):

$$\lim_{m \to \infty} \sum_{n=0}^{\infty} |a_{m,n} - a_{m,n+1}| = 0.$$

Also, using the same definitions for $\alpha_n\{Z(k)\}$, we get:

$$|\alpha_n\{Z(k+h)\overline{Z(k)}\}|^r \leqq \frac{1}{P_n^r} \sum_{k=0}^{n} p_k^r \to 0$$

as $n \to \infty$ for some $1 \leqq r < \infty$, by condition (9).

Hence following the same steps as above we get:

$$\lim_{n \to \infty} \frac{1}{P_n} \sum_{k=0}^{n} p_k e^{2\pi i k^2 \theta} = 0$$

This proves the required result.

As an example of a regular Riesz mean (R, p_n) satisfying the above conditions is the method (R, p_n) defined by $p_n = e^{\log^2 n}$ for all $n = 1, 2, 3, \ldots$. In this case r in condition (9) can be any number > 1. On the other hand [6], this Riesz mean has all summability function $o(n/\log n)$ and does not sum all bounded $(C, 1)$-summable sequences.

References

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