

Gauss and Codazzi equations in a subspace of a Kähler space

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1. Preliminaries

In this paper we shall obtain Gauss and Codazzi equations in a subspace of a Kähler space. Let us consider a Kähler space K_{2n} covered by a system of complex co-ordinate neighbourhoods.

$$(1.1) \quad z^x = \zeta^x + i\zeta^{\bar{x}} \quad *)$$

and a subspace K_{2m} ($m < n$) defined by

$$(1.2) \quad z^x = \zeta^x(\eta^a) + i\zeta^{\bar{x}}(\eta^a) \quad **)$$

where η^a are parameters on the subspace.

Introduce a complex co-ordinate system (u^x) defined by

$$(1.3) \quad u^x = \eta^x + i\eta^{\bar{x}}$$

Representing both the Kähler space K_{2n} and the analytic subspace K_{2m} by the real co-ordinates ζ^h and η^a respectively, we have

$$(1.4) \quad \zeta^h = \zeta^h(\eta^a)$$

The tensor $'F_b^a$ is such that

$$(1.5) \quad F_i^h B_b^i = 'F_b^a B_a^h$$

where $B_b^h = \partial_b \zeta^h$

and

$$(1.6) \quad 'F_c^b 'F_b^a = -A_c^a$$

The induced fundamental metric $'g_{ab}$ is given by

$$(1.7) \quad 'g_{cb} = B_c^j B_b^i 'g_{ji}$$

and

$$(1.8) \quad 'g_{cb} = 'F_c^f 'F_b^e 'g_{fe}$$

*) The notations used in this paper are from Yano's book

**) The indices $a, b, c, \dots$ take values from $1, 2, \dots, m$, $\bar{1}, \bar{2}, \dots, \bar{m}$, and $\alpha, \beta, \dots = 1, 2, \dots, m$.

The Waerden—Bortolotti derivative of B_b^h is given by ([1], p. 103)

$$(1.9) \quad \nabla_c B_b^h = \partial_c B_b^h + B_c^j B_b^i \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} - B_a^h \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\}$$

which is a vector of K_{2n} , orthogonal to K_{2m} .

Let us denote $2n-2m$ mutually orthogonal unit normals to K_{2m} by C_x^h , where

$$x, y, z = m+1, \dots, n, \overline{m+1}, \dots, \overline{n}.$$

Thus ([1], p. 108)

$$(1.10) \quad \nabla_c B_b^h = H_{cbx} C_x^h$$

where H_{cbx} are the second fundamental tensors of K_{2m} with respect to the normals C_x^h . C_x^h satisfies the following relations

$$(1.11) a \quad B_c^j C_x^i g_{ji} = 0,$$

$$(1.11) b \quad C_y^j C_x^i g_{ji} \delta_{yx}.$$

The equation of Weingarten are given by ([1], p. 106)

$$(1.12) \quad \nabla_c C_x^h = -H_{cx}^a B_a^h + L_{cxy} C_y^h,$$

where

$$L_{cxy} = (\nabla_c C_x^j) C_y^i g_{ji}.$$

2. Gauss and Codazzi Equations

Taking Waerden—Bortolotti derivative of (1.9) we get

$$(2.1) \quad \begin{aligned} \nabla_d \nabla_c B_b^h &= \nabla_d (\partial_c B_b^h) + \nabla_d \left(B_c^j B_b^i \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} \right) - \nabla_d B_a^h \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} = \\ &= \partial_d \partial_c B_b^h + \partial_c B_b^i B_d^f \left\{ \begin{matrix} h \\ f \ i \end{matrix} \right\} - \partial_l B_b^h \left\{ \begin{matrix} l \\ d \ c \end{matrix} \right\} - \partial_c B_l^h \left\{ \begin{matrix} l \\ d \ b \end{matrix} \right\} + \\ &\quad + \left[B_c^j (\partial_d B_b^i) \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} + B_b^i (\partial_d B_c^j) \left\{ \begin{matrix} h \\ i \ j \end{matrix} \right\} + B_c^j B_b^i \partial_d \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} + \right. \\ &\quad \left. + B_c^j B_b^i \left\{ \begin{matrix} l \\ j \ i \end{matrix} \right\} \left\{ \begin{matrix} h \\ l \ d \end{matrix} \right\} - B_l^j B_b^i \left\{ \begin{matrix} h \\ i \ j \end{matrix} \right\} \left\{ \begin{matrix} l \\ c \ d \end{matrix} \right\} - B_c^j B_l^i \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} \left\{ \begin{matrix} l \\ b \ d \end{matrix} \right\} \right] - \\ &\quad - \left[\partial_d B_a^h \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} + \partial_d \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} B_a^h + B_a^i B_d^f \left\{ \begin{matrix} h \\ f \ i \end{matrix} \right\} \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} - \right. \\ &\quad \left. - B_a^h \left\{ \begin{matrix} a \\ l \ b \end{matrix} \right\} \left\{ \begin{matrix} l \\ d \ c \end{matrix} \right\} - B_a^h \left\{ \begin{matrix} a \\ c \ l \end{matrix} \right\} \left\{ \begin{matrix} l \\ b \ d \end{matrix} \right\} \right]. \end{aligned}$$

In (2.1), on changing the indices c, d and on subtracting it from the new equation, we obtain the following on simplification

$$(2.2) \quad \nabla_d \nabla_c B_b^h - \nabla_c \nabla_d B_b^h = B_a^h R_{dcb}^a + B_b^i B_c^j K_{jdi}^h - B_b^i S_{cjd}^h,$$

where we have put

$${}^{\prime}R^a_{dc b} = \partial_c \left\{ \begin{matrix} a \\ d \ b \end{matrix} \right\} - \partial_d \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} + \left\{ \begin{matrix} a \\ c \ l \end{matrix} \right\} \left\{ \begin{matrix} l \\ b \ d \end{matrix} \right\} - \left\{ \begin{matrix} a \\ d \ l \end{matrix} \right\} \left\{ \begin{matrix} l \\ b \ c \end{matrix} \right\},$$

$$K_{jdi}^h = \partial_d \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ d \ i \end{matrix} \right\} + \left\{ \begin{matrix} h \\ l \ d \end{matrix} \right\} \left\{ \begin{matrix} l \\ j \ i \end{matrix} \right\} - \left\{ \begin{matrix} h \\ j \ l \end{matrix} \right\} \left\{ \begin{matrix} l \\ d \ i \end{matrix} \right\},$$

and

$$S_{cjd}^h \stackrel{\text{def}}{=} B_c^j \partial_j \left\{ \begin{matrix} h \\ d \ i \end{matrix} \right\} + B_c^j \left\{ \begin{matrix} h \\ j \ l \end{matrix} \right\} \left\{ \begin{matrix} l \\ d \ i \end{matrix} \right\} + B_d^j \partial_c \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} + B_d^j \left\{ \begin{matrix} l \\ j \ i \end{matrix} \right\} \left\{ \begin{matrix} h \\ l \ c \end{matrix} \right\}.$$

Waerden—Bortolotti derivative of (1. 10) yields

$$(2.3) \quad {}^{\prime}\nabla_d {}^{\prime}\nabla_c B_b^h = H_{cbx} {}^{\prime}\nabla_d C_x^h + C_x^h {}^{\prime}\nabla_d H_{cbx}.$$

By changing c, d and subtracting it from the resulting equation so obtained, we get

$$(2.4) \quad {}^{\prime}\nabla_d {}^{\prime}\nabla_c B_b^h - {}^{\prime}\nabla_c {}^{\prime}\nabla_d B_b^h = C_x^h ({}^{\prime}\nabla_d H_{cbx} - {}^{\prime}\nabla_c H_{dbx}) + H_{cbx} {}^{\prime}\nabla_d C_x^h - H_{dbx} {}^{\prime}\nabla_c C_x^h.$$

Using (1. 12), (2. 4) reduces to

$$(2.5) \quad \begin{aligned} & {}^{\prime}\nabla_d {}^{\prime}\nabla_c B_b^h - {}^{\prime}\nabla_c {}^{\prime}\nabla_d B_b^h = \\ & = C_x^h ({}^{\prime}\nabla_d H_{cbx} - {}^{\prime}\nabla_c H_{dbx}) - B_a^h (H_{cbx} H_{dx}^a - H_{dbx} H_{cx}^a) + C_y^h (H_{cbx} L_{dxy} - H_{dbx} L_{cxy}). \end{aligned}$$

From (2. 2) and (2. 5) we get

$$(2.6) \quad \begin{aligned} & B_a^h {}^{\prime}R^a_{dc b} + B_b^i B_c^j K_{jdi}^h - B_b^i S_{cjd}^h = \\ & = C_x^h ({}^{\prime}\nabla_d H_{cbx} - {}^{\prime}\nabla_c H_{dbx}) - B_a^h (H_{cbx} H_{dx}^a - H_{dbx} H_{cx}^a) + C_y^h (H_{cbx} L_{dxy} - H_{dbx} L_{cxy}). \end{aligned}$$

Multiplying both sides of (2. 6) by $g_{kh} B_e^k$ and using (1. 7) and (1. 11)a, we obtain

$$(2.7) \quad {}^{\prime}g_{ea} ({}^{\prime}R^a_{dc b} + H_{cbx} H_{dx}^a - H_{dbx} H_{cx}^a) + B_b^i B_c^j B_e^k g_{kh} K_{jdi}^h - B_b^i B_e^k g_{kh} S_{cjd}^h = 0$$

or

$$(2.8) \quad {}^{\prime}R_{dcbe} = H_{dbx} H_{cxe} - H_{cbx} H_{dxe} - B_b^i B_c^j K_{jdi}^h + B_b^i B_e^k S_{cjd}^h.$$

(2. 8) will be called the “generalised Gauss characteristic equation” in a subspace of a Kähler space.

Again multiplying (2. 6) by $g_{kh} C_z^k$ and using (1. 11) a, b we get

$$(2.9) \quad \delta_{xz} ({}^{\prime}\nabla_d H_{cbx} - {}^{\prime}\nabla_c H_{dbx}) + \delta_{yz} (H_{cbx} L_{dxy} - H_{dbx} L_{cxy}) = 0$$

We call this the “generalised Mainardi Codazzi relation” in a subspace of a Kähler space.

3. Some properties of the second fundamental tensor H_{cbx} .

Let a holomorphic plane element determined by a certain vector u^h and its transform $F_i^h u^i$ by F_i^h at a point be given. We draw a totally geodesic subspace which is tangent to this holomorphic plane element and passes through this point.

We represent the subspace by

$$(3.1) \quad \xi^h = \xi^h(\eta^a)$$

where $a, b, c, d=1, 2$.

Then the subspace being totally geodesic, we have (1, p. 107)

$$(3.2) \quad \partial_c B_b^h + B_c^j B_b^i \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} - B_a^h \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} = 0$$

substituting (3.2) in (1.9), we obtain

$$(3.3) \quad {}'\nabla_c B_b^h = 0$$

from which we have

Theorem 3.1. *A necessary and sufficient condition for B_b^h in a subspace of a Kähler space to be totally geodesic is that its Waerden—Bortolotti derivative be zero*

Applying (3.3) in (2.5) we get

$$(3.4) \quad 0 = C_x^h ({}'\nabla_d H_{cbx} - {}'\nabla_c H_{dbx}) - B_a^h (H_{cbx} H_{dx}^a - H_{dbx} H_{cx}^a) + C_y^h (H_{cbx} L_{dxy} - H_{dbx} L_{cxy}).$$

Multiplying both sides of (3.4) by $g_{kh} B_e^k$ and using (1.7) and (1.11)a we obtain

$${}'g_{ea} (H_{cbx} H_{dx}^a - H_{dbx} H_{cx}^a) = 0$$

or

$$(3.5) \quad H_{cbx} H_{dxe} - H_{dbx} H_{cxe} = 0$$

Thus we have

Theorem 3.2. (3.5) is the necessary condition that a subspace of a Kähler space be totally geodesic.

From (1.11) and (3.3) we get

$$(3.6) \quad H_{cbx} = 0$$

since we assume that $C_x^h \neq 0$.

Thus we have

Theorem 3.3. *A necessary and sufficient condition for a subspace of a Kähler space to be totally geodesic is given by*

$$H_{cbx} = 0$$

Acknowledgments: We are thankful to Dr. M. D. UPADHYAY, for his valuable guidance in the preparation of this paper.

Reference

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(Received October 7, 1970.)