Some remarks on non-abelian homological algebra

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§ 1. Introduction

The aim of these notes is to indicate how one can translate the non-abelian homological algebra of groups over near rings in abstract terms as proposed by FRÖCHLICH at the end of the introduction of his paper [1]. This is of similar nature as done by BUCHSBAUM [2] in translating homological algebra of modules in terms of an exact category; later on this study was continued by HELLER [7]. For our purpose, we shall choose the same category $\mathscr{C}$ as in [4] and continue our study in $\mathscr{C}$. To recall, briefly, $\mathscr{C}$ is a category equipped with the following axioms:

C_1: *$\mathscr{C}$ has a null object.*

C_2: *Every morphism α in $\mathscr{C}$, admits a factorisation as in the diagram.*

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "\nu"'] & & B \\ & L \arrow[ur, "\mu"'] & \end{tikzcd}$$

i.e. $\alpha = \nu\mu$, where ν is a normal epimorphism and μ is a monomorphism

C_3: *$\mathscr{C}$ has product and coproduct for any arbitrary family of objects.*

C_4: *The subobjects and normal factor objects of any object form a set.*

C_5: *If α is a monomorphism and β is a normal epimorphism such that $\alpha\beta$ admits image $\nu'\mu'$, then*

(i) *α normal implies μ' is normal*

(ii) *If (K, μ) denotes the kernel of β, then $(K, \mu) \leqq (A, \alpha)$ and μ' normal will together imply that α is normal.*

Under these axioms, the theory of commutators is available [4]. We shall be frequently using the results and notations of [4] in the sequel. The theory of derived functors and satellites seems plausible and will be left for subsequent study. Various other

*) Contents of this paper form a portion of the author's doctoral thesis at *London University* in 1965. The author understands that some of these results have also been obtained by P. LECOUTURIER in a Hofmannian category; [6].

authors [see SULIŃSKI [8], WIEGANDT and SZÁSZ [9]] have also sudied similar categories for different purposes.

Since our axioms guarantee the existence of *kernels, cokernels* and *normal images*, the concept of *exact* sequences is available as usual [cf. § 8. of 10]. In particular we shall say a short exact sequence

$$0 \to C \xrightarrow{\alpha} B \xrightarrow{\beta} A \to 0 \quad (\mathrm{A})$$

is *central*, when α is a *central monomorphism.*

§ 2. Object pairs; distinguished monomorphisms, epimorphisms and equivalences

By a *pair* $A|(A', \mu')$ we shall mean an object A with a central monomorphism $\mu': A' \to A$; when no confusion arises, we shall indicate a pair by $A|A'$. One observes that A' belongs to the abelian subcategory $\mathscr{A}$ of $\mathscr{C}$.

By a *morphism* $f|f': A|A' \to C|C'$ *of the pairs* we mean a pair of morphisms $f: A \to C, f': A' \to C'$ making the diagram

$$\begin{array}{ccc} A' & \xrightarrow{f'} & C' \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & C \end{array}$$

commutative, the verticals being the usual monomorphisms of the pairs. With the usual composition law, the pairs form a category, which we denote in the sequel by $\mathscr{C}^{(2)}$.

Now any morphism $f: A \to C$ gives rise to a morphism of pairs $f|f': A|A' \to C|C'$ if and only if the image of $\mu' f$ factors through C', where $\mu': A' \to A$ is again the natural monomorphism of the pair $A|A'$.

Thus if $A' \sim 0$, we have a morphism of pairs $A|0 \to C|C'$ in $\mathscr{C}^{(2)}$, for any morphism $f: A \to C$ in $\mathscr{C}$ and this we denote by $f|\omega_{0C'}$.

We denote the morphism $f|\omega: A|0 \to B|0$ induced by f in $\mathscr{C}$ by f again. The identity $1_A: A \to A$ induces a morphism

$$\lambda_{A|A'}: A|0 \to A|A'$$

in $\mathscr{C}^{(2)}$, such that the following diagram

$$\begin{array}{ccc} A|0 & \xrightarrow{\lambda_{A|A'}} & A|A' \\ {\scriptstyle f|\omega}\downarrow & & \downarrow{\scriptstyle f|f'} \\ C|0 & \xrightarrow[\lambda_{C|C'}]{} & C|C' \end{array}$$

commutes.

A morphism pair $f|f': A|A' \to B|B'$ is a *distinguished monomorphism* in $\mathscr{C}^{(2)}$ if f and hence f' is a monomorphism in $\mathscr{C}$. A morphism pair $g|g': A|A' \to B|B'$ is called a *distinguished epimorphism* in $C^{(2)}$, if g and g′ are normal epimorphsms and if $(K, \bar{\mu})$ is the kernel of g, then $\bar{\mu}$ admits a factorisation $\bar{\mu} = \lambda\mu'$ for the pair (A', μ')

From now on *we shall denote by monomorphisms and epimorphisms, in* $\mathscr{C}^{(2)}$ *only to mean in the distinguished sense*. The monomorphisms, epimorphisms in the distinguished sense are indeed monomorphisms, epimorphisms respectively in the usual sense.

We notice that $f|f'$ is invertible if and only if it is a monomorphism and an epimorphism.

A *central sequence of pairs in* $\mathscr{C}^{(2)}$ is a sequence

$$0|0 \to C|C \xrightarrow{f|f'} B|B' \xrightarrow{g|g'} A|A' \to 0|0 \quad \text{(B)}$$

whose component sequences of objects in $\mathscr{C}$ are *central*. Thus (B) can be written in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "f'"] \arrow[d, equal] & B' \arrow[r, "g'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C \arrow[r, "f"'] & B \arrow[r, "g"'] & A \arrow[r] & 0 \end{tikzcd}$$

with central row in $\mathscr{C}$. The bottom row is the *underlying central sequence of objects*.

Now every central sequence of objects in $\mathscr{C}$,

$$0 \to C \xrightarrow{f} B \xrightarrow{g} A \to 0 \quad \text{(C)}$$

can be embeded in a diagram of the form (B) and thus defines a central sequence

$$0|0 \to C|C \xrightarrow{f|1} B|C \xrightarrow{g|\omega_{C0}} A|0 \to 0|0.$$

It is clear that (B) is central in $\mathscr{C}^{(2)}$ if and only if $f|f'$ *is a monomorphism*, $g|g'$ *an epimorphism and*

$$(C, f) = \text{kernel } g.$$

Proposition 2. 1. *Every epimorphism* $g|g': B|B' \to A|A'$ *in* $\mathscr{C}^{(2)}$ *can be extended to a central sequence.*

PROOF. If $(K, \bar{\mu})$ denotes the kernel of g, then $\bar{\mu} = \lambda\mu'$ where $\mu': B' \to B$ is the natural monomorphism of the pair.

Hence

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\lambda"] \arrow[d, equal] & B' \arrow[r, "g'"] \arrow[d, "\mu'"] & A' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r, "\bar{\mu}"'] & B \arrow[r, "g"'] & A \arrow[r] & 0 \end{tikzcd}$$

is a central sequence. Also $\bar{\mu}$ and λ are central [cf. corollary of Proposition 3. 1. 10 of [4]].

Lemma 2. 2. *If the natural monomorphisms* $\mu': B' \to B$ *and* $\mu'': C' \to C$ *admit cokernels* (s, E) and (t, F), *and* $g|g': B|B' \to C|C'$ *is an epimorphism, then* $E \sim F$.

Proof. We consider the diagram

$$\begin{tikzcd}
& B' \arrow[r, "g'"] \arrow[d, "\mu'"] & C' \arrow[d, "\mu''"] \\
K \arrow[ur, "\lambda"] \arrow[r, "\mu"'] & B \arrow[r, "g"'] \arrow[d, "s"] & C \arrow[d, "t"] \\
& E \arrow[r, "\theta'"'] & F
\end{tikzcd}$$

in which

$$(K, \mu) = \text{kernel of } g.$$

Now $\mu' g t = g' \mu'' t = \omega$ implies $gt = s\theta$, for some θ.

Since by definition $\mu = \lambda\mu'$, we have $s = gh$ for some h. Now $g' \mu'' h = \mu' g h = \omega$ which implies $\mu'' h = \omega$, i.e. $h = t\theta'$.

Now $s\theta\theta' = s$ which implies $\theta\theta' = 1$ i.e. θ is a monomorphism. Since it is already a normal epimorphism, it is indeed an equivalence.

§ 3. Results concerning projectives:

A pair $A|A'$ is said to be *$\mathscr{C}^{(2)}$-projective* if every diagram

$$\begin{tikzcd}
& A|A' \arrow[d, "\alpha|\alpha'"] \\
B|B' \arrow[r, "\theta|\theta'"'] & C|C'
\end{tikzcd}$$

of pairs whose row is an epimorphism in $\mathscr{C}^{(2)}$ can be completed to a commutative diagram

$$\begin{tikzcd}
& A|A' \arrow[dl] \arrow[d, "\alpha|\alpha'"] \\
B|B' \arrow[r, "\theta|\theta'"'] & C|C'
\end{tikzcd}$$

Proposition 3. 1. *If A is $\mathscr{C}$-projective, then $A|A'$ is $\mathscr{C}^{(2)}$-projective.*

PROOF. Consider the diagram (D) in $\mathscr{C}^{(2)}$ with $\theta|\theta'$ an epimorphism.
Since A is $\mathscr{C}$-projective, there exists $\varrho: A \to B$ such that $\varrho\theta = \alpha$. Hence $\mu'\varrho\theta = \mu'\alpha = \alpha'\mu''$, where the morphisms involved can be seen in the diagram

$$\begin{tikzcd}
& & A' \arrow[dll, "\varrho'"'] \arrow[dl, "\alpha'"] \arrow[d, "\mu'"] \\
B' \arrow[r, "\theta'"] \arrow[d, "\mu"'] & C' \arrow[d, "\mu''"] & A \arrow[dll, "\varrho"'] \arrow[dl, "\alpha"] \\
B \arrow[r, "\theta"] \arrow[d, "s"'] & C \arrow[d, "t"] & \\
E \arrow[r, "g"] & F &
\end{tikzcd}$$

Now if μ and μ'' admit cokernels (s, E) and (t, F), then by lemma 2. 2 there exists an equivalence $g: E \sim F$, such that

$$sg = \theta t.$$

Now $\mu'\varrho\theta t = \mu'\alpha t = \alpha'\mu'' t = \omega$ which implies $\mu'\varrho s g = \omega$.
i.e. $\mu'\varrho s = \omega$. Since $\mu = \text{kernel}\, s$, $\mu'\varrho = \varrho'\mu$ and for this ϱ', $\varrho'\theta'\mu'' = \alpha'\mu''$ i.e. $\varrho'\theta' = \alpha'$.
Hence $\varrho|\varrho': A|A' \to B|B'$ is the required morphism, such that

$$(\varrho|\varrho')(\theta|\theta') = \alpha|\alpha'.$$

Proposition 3. 2. *$A|(A, 1)$ is $\mathscr{C}^{(2)}$-projective if and only if A is $\mathscr{A}$-projective.*

Proposition 3. 3. *If $\mathscr{C} = \mathscr{A}$, the category of abelian objects, then $A|A'$ is $\mathscr{A}^{(2)}$-projective, if and only if A is $\mathscr{A}$-projective.*

PROOF. By proposition 3. 1, if A is $\mathscr{A}$-projective, then $A|A'$ is $\mathscr{A}^{(2)}$-projective. Conversely, if $A|A'$ is $\mathscr{A}^{(2)}$-projective, then for any diagram

$$\begin{tikzcd}
& A \arrow[d, "\alpha"] & \\
B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

with row an epimorphism, we have an induced diagram

$$\begin{tikzcd}
& A|A' \arrow[d, "\alpha|\mu'\alpha"] & \\
B|B \arrow[r, "\theta|\theta"'] & C|C \arrow[r] & 0|0
\end{tikzcd}$$

in $\mathscr{A}^{(2)}$ for the pair $A|(A', \mu')$. Thus the assertion follows from the fact that $A|A'$ is $\mathscr{A}^{(2)}$-projective.

Proposition 3. 4. *If $A|A'$ and $C|C$ are $\mathscr{C}^{(2)}$-projective, then so is $A\times C|A'\times C$.*

Proof. For the pair $A|(A', \mu')$, $\mu'\times 1: A'\times C \to A\times C$ is central, [cf. Proposition 3. 1. 9 of [4]]. Thus we have the pair $A\times C|(A'\times C, \mu'\times 1)$.

Next let $g|g': D|D' \to E|E'$ be an epimorphism in $\mathscr{C}^{(2)}$ and $f|f': A\times C|A'\times C \to$ $\to E|E'$ a morphism. Then we have commutative diagrams

$$\begin{array}{ccc} A' & \xrightarrow{\sigma_1'} & A'\times C \\ {\scriptstyle \mu'}\downarrow & & \downarrow{\scriptstyle \mu'\times 1} \\ A & \xrightarrow[\sigma_1]{} & A\times C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\sigma_2'} & A'\times C \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \mu'\times 1} \\ C & \xrightarrow[\sigma_2]{} & A\times C \end{array}$$

where the horizontals are the monomorphisms of the products [cf. Lemma 3. 1. 3 of [4]]; i.e. $\sigma_1|\sigma_1'$ and $\sigma_2|\sigma_2'$ are morphisms in $\mathscr{C}^{(2)}$. Hence $\sigma_1 f|\sigma_1' f'$ determines a $\lambda|\lambda'$ such that

$$\lambda g|\lambda' g' = \sigma_1 f|\sigma_1' f'.$$

Similarly there exists $\varrho|\varrho': C|C \to D|D'$ such that

$$\sigma g|\sigma' g' = \sigma_2 f|\sigma_2' f'.$$

Now since $\varrho': C \to D'$ and $\lambda': A \to D'$ are central, (D' being abelian) they determine a unique

$$\lambda' \circ \varrho',$$

such that

$$\sigma_1'(\lambda' \circ \varrho') = \lambda'; \quad \sigma_2'(\lambda' \circ \varrho') = \varrho'.$$

Also $\varrho = \varrho' \mu^*$ is central. [Corollary 3, Proposition 3. 1. 2 of [4]]. Thus λ and ϱ commute and infact $\lambda \circ \varrho = (\lambda \times 1)(1 \circ \varrho)$ [cf. Proposition 3. 1. 12 of [4]]. and so $\lambda \circ \varrho|\lambda' \circ \varrho': A\times C|A'\times C \to D|D'$ is a morphism in $\mathscr{C}^{(2)}$, since

$$\theta = (\mu' \times 1)(\lambda \circ \varrho) = (\lambda' \circ \varrho')\mu^*$$

as follows from the uniqueness of θ, determined by the components $\sigma_1'\theta$, $\sigma_2'\theta$.

Similarly $(\lambda \circ \varrho)g = f$, $(\lambda' \circ \varrho')g' = f'$.

Hence $A\times C|A'\times C$ is $\mathscr{C}^{(2)}$-projective.

Proposition 3. 5. *If (B, μ) is a normal subobject of A, and $\mu^* = (\mu, 1_A)$ is the commutator ideal of the morphisms μ and 1_A, then $\mu^* = \sigma\mu$. If μ^* and σ admit cokernels (ε, F) and (ϱ, D), then there exists a monomorphism $\beta: D \to F$ such that $F|D$ is a pair. if A is $\mathscr{C}$-projective, then $F|D$ is $\mathscr{C}^{(2)}$-projective.*

Proof. First part is essentially proposition 4. 1. 5 of [4]. For the second part, since $\sigma\mu\varepsilon = \mu^*\varepsilon = \omega$, we have $\mu\varepsilon = \varrho\beta$. We shall now check that β is a central monomorphism. Let β admit image $\beta = \delta\varkappa$ and let (L, λ) be the kernel of $\varrho\delta$, then

$$\sigma = \theta\lambda \quad \text{for some} \quad \theta.$$

Also $\lambda\mu\varepsilon = \omega$ implies $\lambda\mu = \Phi\mu^*$. Thus $\lambda\mu = \Phi\mu^* = \Phi\theta\lambda\mu$ which implies $\Phi\theta = 1$. i.e. θ is a retraction hence a normal epimorphism and therefore invertible.

Thus (C, σ) serves as the kernel of $\varrho\delta$ i.e. $\varrho\delta$ and ϱ both serve as cokernels of σ. Hence δ is in an equivalence, showing that β is a monomorphism.

Now since $[\mu, 1_A]=\varepsilon$ is a commutator quotient, $\mu\varepsilon$ and ε commute i.e. $\varrho\beta$ and ε commute.

So β and 1_F commute by Proposition 3. 1. 4 of [4] i.e. β is central.

Next let $g|g':M|M' \to L|L'$ and

$$f|f':F|D \to L|L'$$

be an epimorphism and a morphism respectively in $\mathscr{C}^{(2)}$. Now since A is $\mathscr{C}$-projective, there exists η, such that

$$\eta g = \varepsilon f.$$

Then as in the proof of Proposition 3. 1, this η determines a $\eta':B \to M'$ such that $\eta'\bar{\mu}=\mu\eta$ where $\bar{\mu}:M' \to M$ is the natural monomorphism.

Since $\eta'\bar{\mu}$ is central, it commutes with η and since $[\mu, 1_A]=\varepsilon$ we must have $\eta=\varepsilon\xi$ for some ξ.

Again

$$\eta g=\varepsilon\xi g=\varepsilon f \quad \text{which implies} \quad \xi g=f.$$

Again as before this ξ determines a ξ', such that

$$\beta\xi = \xi'\bar{\mu}.$$

Thus $\xi|\xi':F|D \to M|M'$ such that

$$(\xi|\xi')(g|g') = f|f'.$$

Next we assume our category $\mathscr{C}$ has the further additional axiom;*) C_6. *C has preserving pull backs and push outs.* By this we mean in the pull back diagram

$$\begin{array}{ccc} P & \xrightarrow{\beta_2} & A_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ A_1 & \xrightarrow[\alpha_1]{} & A \end{array}$$

if α_1 is a normal epimorphism so is β_2 and dual considerations holds for monomorphisms in the push outs.

Proposition 3. 6. *If $\mathscr{C}$ has enough projectives so has $\mathscr{C}^{(2)}$.*

Proof. Suppose $\mathscr{C}$ has enough projectives, and consider the pair $A|A'$. Then there exists a projective object $\bar{B}$, with a normal epimorphism $\bar{h}:\bar{B} \to A$. Let P be the inverse image of A' i.e. consider the pull back diagram

$$\begin{array}{ccc} P & \xrightarrow{h'} & A' \\ {\scriptstyle\bar{\mu}}\downarrow & & \downarrow{\scriptstyle\delta} \\ \bar{B} & \xrightarrow[\bar{h}]{} & A \end{array}$$

in which h' is a normal epimorphism; then $(P, \bar{\mu})$ is a subobject of $\bar{B}$. We notice

*) For our purpose, we are using much weaker form of this axiom namely

(i) Existence of normal inverse images in Proposition 3.6 and

(ii) If the monomorphisms μ, $\mu\theta$ admit cokernels ε and ε', then if θ is a monomorphism, so is the induced morphism θ', for which $\theta\varepsilon=\varepsilon\theta'$ in the proof of Theorem 4.1.

that $\bar{\mu}$ is in fact a normal monomorphism. For if (K, μ) is the kernel of $\bar{h}$, then there exists a unique $\theta: K \to P$, such that $\theta\bar{\mu}=\mu$ and $\theta h'=\omega$, showing $(K, \mu)\equiv(P, \bar{\mu})$; also $\bar{h}$ is a normal epimorphism and $\bar{\mu}\bar{h}=h'\delta$ is central as such has normal image. Thus by axiom C_5(ii) $\bar{\mu}$ is normal Now if $\mu^*=(\bar{\mu}, 1_B)$. Then $\mu^*=\lambda\bar{\mu}$. Thus if λ, μ^* admits cokernels (α, D) and (ε^*, F) then there exists a $\beta: D \to F$, such that $F|(D, \beta)$ is a projective pair in $\mathscr{C}^{(2)}$ and $\bar{\mu}\varepsilon^*=\alpha\beta$ for $\alpha: P \to D$ (Proposition 3. 5).

Now since $\mu^*\bar{h}=(\bar{\mu}, 1_{\bar{B}})\bar{h}$

$$=\lambda(\bar{\mu}\bar{h}, \bar{h}) \text{ [cf. Proposition 4. 1. 4 of [4]]}$$

$$=\omega, \text{ since } \bar{\mu}\bar{h} \text{ is central.}$$

Thus there exists a normal epimorphism ϱ, such that $\varepsilon^*\varrho=\bar{h}$. This ϱ induces a normal epimorphism $\varrho': D \to A'$ such that $\alpha\varrho'=h'$. Now $\alpha\varrho'\delta=h'\delta=\bar{\mu}\bar{h}=\alpha\beta\varrho$ implies $\varrho'\delta=\beta\varrho$. We need only to check that, kernel $\varrho\leqq\beta$; to see this we use

Lemma 3. 7. *If v, v' are normal epimorphisms having kernel μ, μ' respectively such that $\mu'\leqq\mu$, then there exists normal epimorphisms α' such that $v'\alpha'=v$ and kernel of α' is the image $(L, \bar{\mu})$ in the canonical decomposition $\mu v'=\bar{v}\bar{\mu}$.*

Now let $\mu\varepsilon^*$ admit image $\hat{v}\hat{\mu}$, then $\hat{\mu}$=kernel ϱ. Thus if $\varkappa$ is the cokernel of β, then

$$\hat{v}\hat{\mu}\varkappa = \mu\varepsilon^*\varkappa = \omega$$

so $\hat{\mu}\varkappa=\omega$ i.e. $\hat{\mu}\leqq\beta$ (since β being central is the kernel of $\varkappa$).

The central sequence (B) *splits* if there exists a morphism $h|h': A|A' \to B|B'$ such that $(g|g')(h|h')=1_A|1_{A'}$. It is easily seen that the central sequence (B) splits if and only if the underlying central sequence of objects splits. From the definition and Propositions 3. 6 and 2. 1, we have

Proposition 3. 8. *$A|A'$ is $\mathscr{C}^{(2)}$-projective if and only if*

(i) *every central sequence (B) splits*

or

(ii) *some central sequences (B) — with $B|B'$ $\mathscr{C}^{(2)}$-projective splits.*

If $\mathscr{B}$ is a variety in $\mathscr{C}$, with associated variety functor V and quotient functor U [5], then for any pair $A|A'$ in $\mathscr{C}$, we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & V(A) \arrow[r, "\mu_A"] & A \arrow[r, "\varepsilon_A"] & U(A) \arrow[r] & 0 \\
& & A' \arrow[u, "\mu"] \arrow[r, "v'"'] & M \arrow[u, "\mu'"'] &
\end{tikzcd}$$

in which the top row is exact, and $\mu\varepsilon_A$ admits image $v'\mu'$. We declare

$$U_2(A|A') = \frac{U(A)}{(M, \mu')}.$$

We denote by $\mathscr{B}^{(2)}$, the full subcategory of $\mathscr{C}^{(2)}$ whose objects are pairs $B|B'$ with $B\in\mathscr{B}$; then we have

Proposition 3. 9. *If $A|A'$ is $\mathscr{C}^{(2)}$-projective, $U_2(A|A')$ is $\mathscr{B}^{(2)}$-projective.*

PROOF. U_2 defined above is a functor from $C^{(2)} \to \mathscr{B}^{(2)}$ which is in fact a left adjoint to the inclusion functor: $\mathscr{B}^{(2)} \to \mathscr{C}^{(2)}$. Hence the assertion follows, since the inclusion functor preserves epimorphisms. In fact U_2 preserves projectiveness defined with respect to usual epimorphisms even, (i.e. not distinguished ones only).

§ 4. Connecting homomorphisms

Let

$$\begin{array}{ccccccc} & A' & \xrightarrow{\alpha'} & A & \xrightarrow{\alpha} & A'' & \to 0 \\ & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & \\ 0 \to & B' & \xrightarrow[\beta']{} & B & \xrightarrow[\beta]{} & B'' & \end{array} \qquad \text{(H)}$$

be a commutative diagram with exact rows. The using similar techniques as in Buchsbaum [[2], Theorem 5. 8], [save for the dual construction we use preserving push out form, see foot note on page 111], we salvage the non-abelian form of his theorem 5. 8 in [2].

Theorem 4. 1. *The diagram (H) gives rise to a sequence of homomorphisms*

$$\operatorname{Ker} f' \to \operatorname{Ker} f \to \operatorname{Ker} f'' \xrightarrow{\delta} \operatorname{Coker} f' \to \operatorname{Coker} f \to \operatorname{Coker} f''.$$

The composition of any two consecutive mappings in this sequence is null. If α' is a monomorphism, then $\operatorname{Ker} f' \to \operatorname{Ker} f$ is a monomorphism. If β is a normal epimorphism, then $\operatorname{Coker} f \to \operatorname{Coker} f''$ is such a epimorphism. The sequence $\operatorname{Ker} f' \to \operatorname{Ker} f \to \operatorname{Ker} f''$ is exact.

It is not difficult to see, how one can translate the familiar facts of $\mathscr{C}^{(2)}$-resolutions and $\mathscr{C}^{(2)}$-representations and studied by Fröhlich in § 5 of [1].

By taking projective resolutions of pairs and using Heller's results [7], one can develop the later part of the theory as mentioned in the introduction and will be left for the future.

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