

Some remarks on non-abelian homological algebra

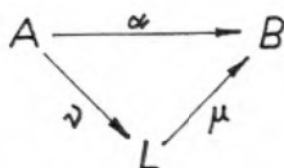
By SYED A. HUQ (Canberra)*)

§ 1. Introduction

The aim of these notes is to indicate how one can translate the non-abelian homological algebra of groups over near rings in abstract terms as proposed by FRÖCHLICH at the end of the introduction of his paper [1]. This is of similar nature as done by BUCHSBAUM [2] in translating homological algebra of modules in terms of an exact category; later on this study was continued by HELLER [7]. For our purpose, we shall choose the same category $\mathcal{C}$ as in [4] and continue our study in $\mathcal{C}$. To recall, briefly, $\mathcal{C}$ is a category equipped with the following axioms:

C_1 : $\mathcal{C}$ has a null object.

C_2 : Every morphism α in $\mathcal{C}$, admits a factorisation as in the diagram.



i.e. $\alpha = \nu\mu$, where ν is a normal epimorphism and μ is a monomorphism

C_3 : $\mathcal{C}$ has product and coproduct for any arbitrary family of objects.

C_4 : The subobjects and normal factor objects of any object form a set.

C_5 : If α is a monomorphism and β is a normal epimorphism such that $\alpha\beta$ admits image $\nu'\mu'$, then

(i) α normal implies μ' is normal

(ii) If (K, μ) denotes the kernel of β , then $(K, \mu) \leq (A, \alpha)$ and μ' normal will together imply that α is normal.

Under these axioms, the theory of commutators is available [4]. We shall be frequently using the results and notations of [4] in the sequel. The theory of derived functors and satellites seems plausible and will be left for subsequent study. Various other

*) Contents of this paper form a portion of the author's doctoral thesis at London University in 1965. The author understands that some of these results have also been obtained by P. LECOUTURIER in a Hofmannian category; [6].

authors [see SULIŃSKI [8], WIEGANDT and SZÁSZ [9]] have also studied similar categories for different purposes.

Since our axioms guarantee the existence of *kernels*, *cokernels* and *normal images*, the concept of *exact* sequences is available as usual [cf. § 8. of 10]. In particular we shall say a short exact sequence

$$0 \rightarrow C \xrightarrow{\alpha} B \xrightarrow{\beta} A \rightarrow 0 \quad (A)$$

is *central*, when α is a *central monomorphism*.

§ 2. Object pairs; distinguished monomorphisms, epimorphisms and equivalences

By a *pair* $A|(A', \mu')$ we shall mean an object A with a central monomorphism $\mu': A' \rightarrow A$; when no confusion arises, we shall indicate a pair by $A|A'$. One observes that A' belongs to the abelian subcategory $\mathcal{A}$ of $\mathcal{C}$.

By a *morphism* $f|f': A|A' \rightarrow C|C'$ of the pairs we mean a pair of morphisms $f: A \rightarrow C, f': A' \rightarrow C'$ making the diagram

$$\begin{array}{ccc} A' & \xrightarrow{f'} & C' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & C \end{array}$$

commutative, the verticals being the usual monomorphisms of the pairs. With the usual composition law, the pairs form a category, which we denote in the sequel by $\mathcal{C}^{(2)}$.

Now any morphism $f: A \rightarrow C$ gives rise to a morphism of pairs $f|f': A|A' \rightarrow C|C'$ if and only if the image of $\mu'f$ factors through C' , where $\mu': A' \rightarrow A$ is again the natural monomorphism of the pair $A|A'$.

Thus if $A' \sim 0$, we have a morphism of pairs $A|0 \rightarrow C|C'$ in $\mathcal{C}^{(2)}$, for any morphism $f: A \rightarrow C$ in $\mathcal{C}$ and this we denote by $f|\omega_{0C'}$.

We denote the morphism $f|\omega: A|0 \rightarrow B|0$ induced by f in $\mathcal{C}$ by f again. The identity $1_A: A \rightarrow A$ induces a morphism

$$\lambda_{A|A'}: A|0 \rightarrow A|A'$$

in $\mathcal{C}^{(2)}$, such that the following diagram

$$\begin{array}{ccc} A|0 & \xrightarrow{\lambda_{A|A'}} & A|A' \\ f|\omega \downarrow & & \downarrow f|f' \\ C|0 & \xrightarrow{\lambda_{C|C'}} & C|C' \end{array}$$

commutes.

A morphism pair $f|f': A|A' \rightarrow B|B'$ is a *distinguished monomorphism* in $\mathcal{C}^{(2)}$ if f and hence f' is a monomorphism in $\mathcal{C}$. A morphism pair $g|g': A|A' \rightarrow B|B'$ is called a *distinguished epimorphism* in $\mathcal{C}^{(2)}$, if g and g' are normal epimorphisms and if $(K, \bar{\mu})$ is the kernel of g , then $\bar{\mu}$ admits a factorisation $\bar{\mu} = \lambda\mu'$ for the pair (A', μ')

From now on we shall denote by *monomorphisms and epimorphisms, in $\mathcal{C}^{(2)}$ only to mean in the distinguished sense*. The monomorphisms, epimorphisms in the distinguished sense are indeed monomorphisms, epimorphisms respectively in the usual sense.

We notice that $f|f'$ is invertible if and only if it is a monomorphism and an epimorphism.

A *central sequence of pairs* in $\mathcal{C}^{(2)}$ is a sequence

$$0|0 \rightarrow C|C \xrightarrow{f|f'} B|B' \xrightarrow{g|g'} A|A' \rightarrow 0|0 \quad (B)$$

whose component sequences of objects in $\mathcal{C}$ are *central*. Thus (B) can be written in a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & C & \xrightarrow{f'} & B' & \xrightarrow{g'} & A' \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & C & \xrightarrow{f} & B & \xrightarrow{g} & A \rightarrow 0 \end{array}$$

with central row in $\mathcal{C}$. The bottom row is the *underlying central sequence of objects*.

Now every central sequence of objects in $\mathcal{C}$,

$$0 \rightarrow C \xrightarrow{f} B \xrightarrow{g} A \rightarrow 0 \quad (C)$$

can be embedded in a diagram of the form (B) and thus defines a central sequence

$$0|0 \rightarrow C|C \xrightarrow{f|1} B|C \xrightarrow{g|\omega_{C0}} A|0 \rightarrow 0|0.$$

It is clear that (B) is central in $\mathcal{C}^{(2)}$ if and only if $f|f'$ is a monomorphism, $g|g'$ an epimorphism and

$$(C, f) = \text{kernel } g.$$

Proposition 2. 1. *Every epimorphism $g|g': B|B' \rightarrow A|A'$ in $\mathcal{C}^{(2)}$ can be extended to a central sequence.*

PROOF. If $(K, \bar{\mu})$ denotes the kernel of g , then $\bar{\mu} = \lambda\mu'$ where $\mu': B' \rightarrow B$ is the natural monomorphism of the pair.

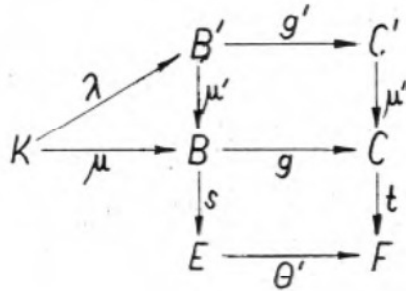
Hence

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \xrightarrow{\lambda} & B' & \xrightarrow{g'} & A' \rightarrow 0 \\ & & \parallel & & \downarrow \mu' & & \downarrow \\ 0 & \rightarrow & K & \xrightarrow{\bar{\mu}} & B & \xrightarrow{g} & A \rightarrow 0 \end{array}$$

is a central sequence. Also $\bar{\mu}$ and λ are central [cf. corollary of Proposition 3. 1. 10 of [4]].

Lemma 2. 2. *If the natural monomorphisms $\mu': B' \rightarrow B$ and $\mu'': C' \rightarrow C$ admit co-kernels (s, E) and (t, F) , and $g|g': B|B' \rightarrow C|C'$ is an epimorphism, then $E \sim F$.*

PROOF. We consider the diagram



in which

$(K, \mu) = \text{kernel of } g$.

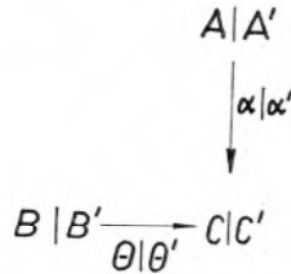
Now $\mu'gt = g'\mu''t = \omega$ implies $gt = s\theta$, for some θ .

Since by definition $\mu = \lambda\mu'$, we have $s = gh$ for some h . Now $g'\mu''h = \mu'gh = \omega$ which implies $\mu''h = \omega$, i.e. $h = t\theta'$.

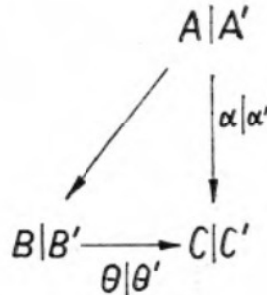
Now $s\theta\theta' = s$ which implies $\theta\theta' = 1$ i.e. θ is a monomorphism. Since it is already a normal epimorphism, it is indeed an equivalence.

§ 3. Results concerning projectives:

A pair $A|A'$ is said to be $\mathcal{C}^{(2)}$ -projective if every diagram



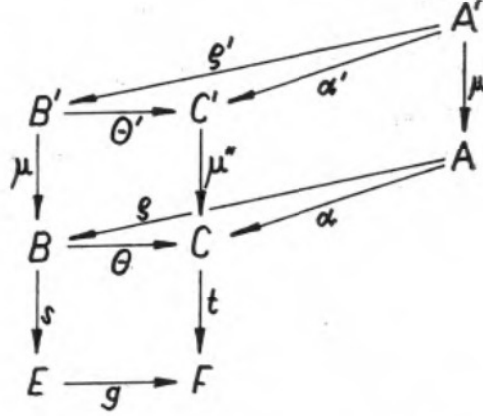
of pairs whose row is an epimorphism in $\mathcal{C}^{(2)}$ can be completed to a commutative diagram



Proposition 3.1. *If A is $\mathcal{C}$ -projective, then $A|A'$ is $\mathcal{C}^{(2)}$ -projective.*

PROOF. Consider the diagram (D) in $\mathcal{C}^{(2)}$ with $\theta|\theta'$ an epimorphism.

Since A is $\mathcal{C}$ -projective, there exists $\varrho:A \rightarrow B$ such that $\varrho\theta=\alpha$. Hence $\mu'\varrho\theta=\mu'\alpha=\alpha'\mu''$, where the morphisms involved can be seen in the diagram



Now if μ and μ'' admit cokernels (s, E) and (t, F) , then by lemma 2.2 there exists an equivalence $g:E \sim F$, such that

$$sg = \theta t.$$

Now $\mu'\varrho\theta t = \mu'\alpha t = \alpha'\mu''t = \omega$ which implies $\mu'\varrho sg = \omega$.

i.e. $\mu'\varrho s = \omega$. Since $\mu = \text{ker } s$, $\mu'\varrho = \varrho'\mu$ and for this ϱ' , $\varrho'\theta'\mu'' = \alpha'\mu''$ i.e. $\varrho'\theta' = \alpha'$.

Hence $\varrho|\varrho': A|A' \rightarrow B|B'$ is the required morphism, such that

$$(\varrho|\varrho')(\theta|\theta') = \alpha|\alpha'.$$

Proposition 3.2. *$A|(A, 1)$ is $\mathcal{C}^{(2)}$ -projective if and only if A is $\mathcal{A}$ -projective.*

Proposition 3.3. *If $\mathcal{C} = \mathcal{A}$, the category of abelian objects, then $A|A'$ is $\mathcal{A}^{(2)}$ -projective, if and only if A is $\mathcal{A}$ -projective.*

PROOF. By proposition 3.1, if A is $\mathcal{A}$ -projective, then $A|A'$ is $\mathcal{A}^{(2)}$ -projective. Conversely, if $A|A'$ is $\mathcal{A}^{(2)}$ -projective, then for any diagram

$$\begin{array}{c} A \\ \downarrow \alpha \\ B \rightarrow C \rightarrow 0 \end{array}$$

with row an epimorphism, we have an induced diagram

$$\begin{array}{c} A|A' \\ \downarrow \alpha|\mu'\alpha \\ B|B \xrightarrow{\theta|\theta} C|C \rightarrow 0|0 \end{array}$$

in $\mathcal{A}^{(2)}$ for the pair $A|(A', \mu')$. Thus the assertion follows from the fact that $A|A'$ is $\mathcal{A}^{(2)}$ -projective.

Proposition 3. 4. *If $A|A'$ and $C|C$ are $\mathcal{C}^{(2)}$ -projective, then so is $A \times C|A' \times C$.*

PROOF. For the pair $A|(A', \mu')$, $\mu' \times 1: A' \times C \rightarrow A \times C$ is central, [cf. Proposition 3. 1. 9 of [4]]. Thus we have the pair $A \times C|(A' \times C, \mu' \times 1)$.

Next let $g|g': D|D' \rightarrow E|E'$ be an epimorphism in $\mathcal{C}^{(2)}$ and $f|f': A \times C|A' \times C \rightarrow E|E'$ a morphism. Then we have commutative diagrams

$$\begin{array}{ccc} A' & \xrightarrow{\sigma'_1} & A' \times C \\ \mu' \downarrow & & \downarrow \mu' \times 1 \\ A & \xrightarrow{\sigma_1} & A \times C \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\sigma'_2} & A' \times C \\ 1 \downarrow & & \downarrow \mu' \times 1 \\ C & \xrightarrow{\sigma_2} & A \times C \end{array}$$

where the horizontals are the monomorphisms of the products [cf. Lemma 3. 1. 3 of [4]]; i.e. $\sigma_1|\sigma'_1$ and $\sigma_2|\sigma'_2$ are morphisms in $\mathcal{C}^{(2)}$. Hence $\sigma_1 f|\sigma'_1 f'$ determines a $\lambda|\lambda'$ such that

$$\lambda g|\lambda' g' = \sigma_1 f|\sigma'_1 f'.$$

Similarly there exists $q|q': C|C \rightarrow D|D'$ such that

$$\sigma g|\sigma' g' = \sigma_2 f|\sigma'_2 f'.$$

Now since $q': C \rightarrow D'$ and $\lambda': A \rightarrow D'$ are central, (D' being abelian) they determine a unique

$$\lambda' \circ q',$$

such that

$$\sigma'_1(\lambda' \circ q') = \lambda'; \quad \sigma'_2(\lambda' \circ q') = q'.$$

Also $q = q' \mu^*$ is central. [Corollary 3, Proposition 3. 1. 2 of [4]]. Thus λ and q commute and infact $\lambda \circ q = (\lambda \times 1)(1 \circ q)$ [cf. Proposition 3. 1. 12 of [4]]. and so $\lambda \circ q|\lambda' \circ q': A \times C|A' \times C \rightarrow D|D'$ is a morphism in $\mathcal{C}^{(2)}$, since

$$\theta = (\mu' \times 1)(\lambda \circ q) = (\lambda' \circ q')\mu^*$$

as follows from the uniqueness of θ , determined by the components $\sigma'_1 \theta, \sigma'_2 \theta$.

Similarly $(\lambda \circ q)g = f, (\lambda' \circ q')g' = f'$.

Hence $A \times C|A' \times C$ is $\mathcal{C}^{(2)}$ -projective.

Proposition 3. 5. *If (B, μ) is a normal subobject of A , and $\mu^* = (\mu, 1_A)$ is the commutator ideal of the morphisms μ and 1_A , then $\mu^* = \sigma\mu$. If μ^* and σ admit cokernels (ε, F) and (q, D) , then there exists a monomorphism $\beta: D \rightarrow F$ such that $F|D$ is a pair. if A is $\mathcal{C}$ -projective, then $F|D$ is $\mathcal{C}^{(2)}$ -projective.*

PROOF. First part is essentially proposition 4. 1. 5 of [4]. For the second part, since $\sigma\mu\varepsilon = \mu^*\varepsilon = \omega$, we have $\mu\varepsilon = q\beta$. We shall now check that β is a central monomorphism. Let β admit image $\beta = \delta\lambda$ and let (L, λ) be the kernel of $q\delta$, then

$$\sigma = \theta\lambda \quad \text{for some } \theta.$$

Also $\lambda\mu\varepsilon = \omega$ implies $\lambda\mu = \Phi\mu^*$. Thus $\lambda\mu = \Phi\mu^* = \Phi\theta\lambda\mu$ which implies $\Phi\theta = 1$. i.e. θ is a retraction hence a normal epimorphism and therefore invertible.

Thus (C, σ) serves as the kernel of $q\delta$ i.e. $q\delta$ and q both serve as cokernels of σ . Hence δ is in an equivalence, showing that β is a monomorphism.

Now since $[\mu, 1_A] = \varepsilon$ is a commutator quotient, $\mu\varepsilon$ and ε commute i.e. $q\beta$ and ε commute.

So β and 1_F commute by Proposition 3.1.4 of [4] i.e. β is central.

Next let $g|g': M|M' \rightarrow L|L'$ and

$$f|f': F|D \rightarrow L|L'$$

be an epimorphism and a morphism respectively in $\mathcal{C}^{(2)}$. Now since A is $\mathcal{C}$ -projective, there exists η , such that

$$\eta g = \varepsilon f.$$

Then as in the proof of Proposition 3.1, this η determines a $\eta': B \rightarrow M'$ such that $\eta' \bar{\mu} = \mu \eta$ where $\bar{\mu}: M' \rightarrow M$ is the natural monomorphism.

Since $\eta' \bar{\mu}$ is central, it commutes with η and since $[\mu, 1_A] = \varepsilon$ we must have $\eta = \varepsilon \xi$ for some ξ .

Again

$$\eta g = \varepsilon \xi g = \varepsilon f \quad \text{which implies} \quad \xi g = f.$$

Again as before this ξ determines a ξ' , such that

$$\beta \xi = \xi' \bar{\mu}.$$

Thus $\xi|\xi': F|D \rightarrow M|M'$ such that

$$(\xi|\xi')(g|g') = f|f'.$$

Next we assume our category $\mathcal{C}$ has the further additional axiom;*) C_6 . $\mathcal{C}$ has preserving pull backs and push outs. By this we mean in the pull back diagram

$$\begin{array}{ccc} P & \xrightarrow{\beta_2} & A_2 \\ \beta_1 \downarrow & & \downarrow \alpha_2 \\ A_1 & \xrightarrow{\alpha_1} & A \end{array}$$

if α_1 is a normal epimorphism so is β_2 and dual considerations holds for monomorphisms in the push outs.

Proposition 3.6. *If $\mathcal{C}$ has enough projectives so has $\mathcal{C}^{(2)}$.*

PROOF. Suppose $\mathcal{C}$ has enough projectives, and consider the pair $A|A'$. Then there exists a projective object $\bar{B}$, with a normal epimorphism $\bar{h}: \bar{B} \rightarrow A$. Let P be the inverse image of A' i.e. consider the pull back diagram

$$\begin{array}{ccc} P & \xrightarrow{h'} & A' \\ \bar{\mu} \downarrow & & \downarrow \delta \\ \bar{B} & \xrightarrow{\bar{h}} & A \end{array}$$

in which h' is a normal epimorphism; then $(P, \bar{\mu})$ is a subobject of $\bar{B}$. We notice

*) For our purpose, we are using much weaker form of this axiom namely

(i) Existence of normal inverse images in Proposition 3.6 and

(ii) If the monomorphisms $\mu, \mu\theta$ admit cokernels ε and ε' , then if θ is a monomorphism, so is the induced morphism θ' , for which $\theta\varepsilon = \varepsilon\theta'$ in the proof of Theorem 4.1.

that $\bar{\mu}$ is in fact a normal monomorphism. For if (K, μ) is the kernel of $\bar{h}$, then there exists a unique $\theta: K \rightarrow P$, such that $\theta\bar{\mu} = \mu$ and $\theta h' = \omega$, showing $(K, \mu) \cong (P, \bar{\mu})$; also $\bar{h}$ is a normal epimorphism and $\bar{\mu}\bar{h} = h'\delta$ is central as such has normal image. Thus by axiom $C_5(ii)$ $\bar{\mu}$ is normal. Now if $\mu^* = (\bar{\mu}, 1_B)$. Then $\mu^* = \lambda\bar{\mu}$. Thus if λ, μ^* admits cokernels (α, D) and (ε^*, F) then there exists a $\beta: D \rightarrow F$, such that $F|(D, \beta)$ is a projective pair in $\mathcal{C}^{(2)}$ and $\bar{\mu}\varepsilon^* = \alpha\beta$ for $\alpha: P \rightarrow D$ (Proposition 3.5).

Now since $\mu^*\bar{h} = (\bar{\mu}, 1_B)\bar{h}$

$$= \lambda(\bar{\mu}\bar{h}, \bar{h}) \text{ [cf. Proposition 4.1.4 of [4]]}$$

$$= \omega, \text{ since } \bar{\mu}\bar{h} \text{ is central.}$$

Thus there exists a normal epimorphism ϱ , such that $\varepsilon^*\varrho = \bar{h}$. This ϱ induces a normal epimorphism $\varrho': D \rightarrow A'$ such that $\alpha\varrho' = h'$. Now $\alpha\varrho'\delta = h'\delta = \bar{\mu}\bar{h} = \alpha\beta\varrho$ implies $\varrho'\delta = \beta\varrho$. We need only to check that, $\text{kernel } \varrho \leq \beta$; to see this we use

Lemma 3.7. *If v, v' are normal epimorphisms having kernel μ, μ' respectively such that $\mu' \leq \mu$, then there exists normal epimorphisms α' such that $v'\alpha' = v$ and kernel of α' is the image $(L, \bar{\mu})$ in the canonical decomposition $\mu v' = \bar{v}\bar{\mu}$.*

Now let $\mu\varepsilon^*$ admit image $\hat{v}\hat{\mu}$, then $\hat{\mu} = \text{kernel } \varrho$. Thus if κ is the cokernel of β , then

$$\hat{v}\hat{\mu}\kappa = \mu\varepsilon^*\kappa = \omega$$

so $\hat{\mu}\kappa = \omega$ i.e. $\hat{\mu} \leq \beta$ (since β being central is the kernel of κ).

The central sequence (B) splits if there exists a morphism $h|h': A|A' \rightarrow B|B'$ such that $(g|g')(h|h') = 1_A|1_{A'}$. It is easily seen that the central sequence (B) splits if and only if the underlying central sequence of objects splits. From the definition and Propositions 3.6 and 2.1, we have

Proposition 3.8. *$A|A'$ is $\mathcal{C}^{(2)}$ -projective if and only if*

(i) *every central sequence (B) splits*

or

(ii) *some central sequences (B) — with $B|B'$ $\mathcal{C}^{(2)}$ -projective splits.*

If $\mathcal{B}$ is a variety in $\mathcal{C}$, with associated variety functor V and quotient functor U [5], then for any pair $A|A'$ in $\mathcal{C}$, we get a diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & V(A) & \xrightarrow{\mu_A} & A & \xrightarrow{\varepsilon_A} & U(A) \rightarrow 0 \\ & & & \mu \uparrow & & & \uparrow \mu' \\ & & & A' & \xrightarrow{v'} & M & \end{array}$$

in which the top row is exact, and $\mu\varepsilon_A$ admits image $v'\mu'$. We declare

$$U_2(A|A') = \frac{U(A)}{(M, \mu')}.$$

We denote by $\mathcal{B}^{(2)}$, the full subcategory of $\mathcal{C}^{(2)}$ whose objects are pairs $B|B'$ with $B \in \mathcal{B}$; then we have

Proposition 3.9. *If $A|A'$ is $\mathcal{C}^{(2)}$ -projective, $U_2(A|A')$ is $\mathcal{B}^{(2)}$ -projective.*

PROOF. U_2 defined above is a functor from $C^{(2)} \rightarrow \mathcal{B}^{(2)}$ which is in fact a left adjoint to the inclusion functor: $\mathcal{B}^{(2)} \rightarrow \mathcal{C}^{(2)}$. Hence the assertion follows, since the inclusion functor preserves epimorphisms. In fact U_2 preserves projectiveness defined with respect to usual epimorphisms even, (i.e. not distinguished ones only).

§ 4. Connecting homomorphisms

Let

$$\begin{array}{ccccccc} A' & \xrightarrow{\alpha'} & A & \xrightarrow{\alpha} & A'' & \rightarrow & 0 \\ \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 \rightarrow B' & \xrightarrow{\beta'} & B & \xrightarrow{\beta} & B'' & & \end{array} \quad (H)$$

be a commutative diagram with exact rows. The using similar techniques as in Buchsbaum [[2], Theorem 5.8], [save for the dual construction we use preserving push out form, see foot note on page 111], we salvage the non-abelian form of his theorem 5.8 in [2].

Theorem 4.1. *The diagram (H) gives rise to a sequence of homomorphisms*

$$\text{Ker } f' \rightarrow \text{Ker } f \rightarrow \text{Ker } f'' \xrightarrow{\delta} \text{Coker } f' \rightarrow \text{Coker } f \rightarrow \text{Coker } f''.$$

The composition of any two consecutive mappings in this sequence is null. If α' is a monomorphism, then $\text{Ker } f' \rightarrow \text{Ker } f$ is a monomorphism. If β is a normal epimorphism, then $\text{Coker } f \rightarrow \text{Coker } f''$ is such a epimorphism. The sequence $\text{Ker } f' \rightarrow \text{Ker } f \rightarrow \text{Ker } f''$ is exact.

It is not difficult to see, how one can translate the familiar facts of $\mathcal{C}^{(2)}$ -resolutions and $\mathcal{C}^{(2)}$ -representations and studied by Fröhlich in § 5 of [1].

By taking projective resolutions of pairs and using Heller's results [7], one can develop the later part of the theory as mentioned in the introduction and will be left for the future.

Bybliography

- [1] A. FRÖHLICH, Non abelian homological algebra I, Derived functors and satellites, *Proc. London Math. Soc.* **11**, (1961), 239—275.
- [2] D. A. BUCHSBAUM, Exact categories and Duality, *Trans. Amer. Math. Soc.*, **80**, (1955), 1—34.
- [3] H. CARTAN and S. EILENBERG, Homological Algebra, *Princeton* (1955).
- [4] S. A. HUQ — Commutator, Nilpotency and Solvability in categories, *Quart. J. Math. Oxford Ser.* (2), **19**, (1968), 363—389.
- [5] S. A. HUQ — Semivarieties and subfunctors of the identity functor, *Pacific J. Math.* **29**, (1969), 303—309.
- [6] F. HOFMANN, Über eine die Kategorie der Gruppen Umfassende Kategorie, *Sitzungsber. Bayer Akad. Wiss. Math. Natur. Kl. S. B.*, (1960), 163—204.
- [7] A. HELLER, Homological algebra in abelian categories, *Ann. of Math.*, **68**, (1958), 484—525.
- [8] A. SULIŃSKI, The Brown-McCoy radicals in categories, *Fund. Math.*, **59**, (1966), 23—41.
- [9] F. SZÁSZ and R. WIEGANDT, On the dualization of sub direct embeddings, *Acta. Math. Acad. Sci. Hungar.*, **20**, (1969), 289—302.
- [10] A. G. KUROSU, A. KH. LIVSHITS and E. G. SCHULGEIFER, Foundations of theory of categories. *Russian Math. Surveys*, **15**, (1960), 1—46.

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