

Syntopogenous structures and complete regularity

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An extensive theory of syntopogenous structures has been developed by Császár [1]. The purpose of this paper is to study the relationship between syntopogenous structures and complete regularity.

A syntopogenous structure gives rise to two topologies in general and so it is natural to associate a bitopological space with a syntopogenous structure. A syntopogenous structure characterizes a particular type of bitopological space; a bitopological space is completely regular iff it is syntopogenizable. This is similar to the result that a symmetric syntopogenous structure characterizes a particular type of topological space — the completely regular one.

The bitopological space of a perfect syntopogenous structure has some special properties.

Let E denote a set. The empty subset of E will be denoted by $\emptyset$ and for a subset A of E we will write cA for the complement of A . If A contains only one element x we will write cx for cA .

Definition 1. Let $\mathcal{J}, \mathcal{J}'$ be two topologies for E . Then the ordered triple $(E, \mathcal{J}, \mathcal{J}')$ is said to be a bitopological space; $\mathcal{J}$ and $\mathcal{J}'$ are called the left and right topologies of this space.

Let $<$ be a topogenous structure on E as defined on page 59 of Császár (1). Denote by $\mathcal{J}$ the family of all subsets T of E such that $x \in T$ implies $cT < cx$; it is then obvious that $\mathcal{J}$ is a topology for E . Take $\mathcal{J}'$ to be the family of all subsets T of E such that $x \in T$ implies $x < T$; then $\mathcal{J}'$ is also a topology for E .

Definition 2. We will call $(E, <, \mathcal{J}, \mathcal{J}')$ the bitopological space, $\mathcal{J}$ the left topology and $\mathcal{J}'$ the right topology of $<$.

But when the context makes the meaning clear we will also denote this space by $(M, <)$ or M .

Denote by k, k' the Kuratowski closure functions respectively for $\mathcal{J}, \mathcal{J}'$. Express composition of functions by juxtaposition; thus ck will denote $c(kA)$ for all subsets A of E . Take $i = ckc, i' = ck'c$; then i and i' are the interior functions for k and k' . We will also write (E, k, k') for the bitopological space $(E, \mathcal{J}, \mathcal{J}')$. When A contains only a single point x we will write kx for kA and $k'x$ for $k'A$.

Theorem 1. Let A be a subset of E . Then

$$iA = \{x : cA < cx\} \quad \text{and} \quad i'A = \{x : x < A\}.$$

PROOF. Let $B = \{x: cA < cx\}$. Then $iA \subset B \subset A$ and so $iA = B$ if $iB = B$. Let $x \in B$. Then $cA < cx$ and so there is $C \subset E$ such that $cA < C < cx$. Now $y \in cC$ implies $C \subset cy$ and so $cA < cy$; hence $y \in B$ which implies $cC \subset B$. Therefore $cB \subset C$ whence $cB < cx$ and so $iB = B$. The other part of the theorem can be proved similarly.

Corollary. $kA = \{xA \nless cx\}$ and $k'A = \{x: x \nless cA\}$.

Theorem 2. $A < B$ iff $kA < i'B$.

PROOF. Let $A < B$. Then $kA \subset B$ since $x \in kA$ implies $x \in B$ for $x \in cB$ implies $B \subset cx$ and so $A < cx$ which is a contradiction. Also $x \in A$ implies $x < B$ and so $x \in i'B$ from which it follows $A \subset i'B$.

Now $A < B$ implies there is $C \subset E$ such that $A < C < B$. Hence $kA \subset C < B$ and so $kA < B$. Also $A < C \subset i'B$ and so $A < i'B$. Thus $A < B$ implies $kA < B$ which in turn implies $kA < i'B$.

The converse is obvious.

Each topogeneous structure $<$ gives rise to a topogenous structure $<'$ defined by $A <' B$ iff $cB < cA$. It is easy to see that the left and right topologies of $<'$ coincide respectively with the right and left topologies of $<$. Hence $<'$ generates no new topologies.

Definition 3. Let $(E, \mathcal{J}, \mathcal{J}')$, $(N, \mathcal{N}, \mathcal{N}')$ be two bitopological spaces and f a function from E to N . Then f is said to be continuous iff $\mathcal{J} - \mathcal{N}$ is continuous and $\mathcal{J}' - \mathcal{N}'$ is continuous.

Definition 4. Let $(E, \mathcal{J}, \mathcal{J}')$ be a bitopological space and F a subset of E . Denote by $\mathcal{N}, \mathcal{N}'$ the relativizations respectively of $\mathcal{J}, \mathcal{J}'$ to N . Then $(N, \mathcal{N}, \mathcal{N}')$ is said to be a subspace of $(E, \mathcal{J}, \mathcal{J}')$.

Let R be the set of all real numbers. Define a quasimetric m for R as follows: for all real x, y ,

$$m'(x, y) = \begin{cases} y - x, & x \leq' y \\ 0, & y <' x \end{cases}$$

where $<'$ denotes the usual order for the reals. For subsets A, B of R write $M(A, B) = \inf \{m'(x, y): x \in A, y \in B\}$. Define $<^*$ for R by $A <^* B$ iff $M(A, cB) > 0$. Then $<^*$ is a topogenous structure on R .

Definition 5. We will call $<^*$ the usual topogenous structure on R and the bitopological space of $<^*$ the usual bitopological space for R . If A is a subset of R then the subspace for A is said to be the usual bitopological space for A . Denote by (R, r, r') the usual bitopological space for R . Let I denote the closed unit interval $[0, 1]$ of the reals. We will also denote by I the usual bitopological space for I .

Definition 6. A bitopological space $(E, \mathcal{J}, \mathcal{J}')$ is said to be completely regular iff

- (i) A is $\mathcal{J}$ -closed and y in cA imply there is a continuous function f from E to I such that $fA = 0$ and $f(y) = 1$ and
- (ii) B is $\mathcal{J}'$ -closed and $x \in cB$ imply there is a continuous function g from E to I such that $g(x) = 0$ and $gB = 1$.

For x, y in R let $xR = \{y: x <' y\}$ and $Rx = \{y: y <' x\}$ where $<'$ is the usual order relation for the reals. Then the set of all xR for x in R is a base for the left topology of R and the set of all Rx for x in R is a base for the right topology of R .

$<'$ will denote the usual order relation for the reals in Lemmas 1, 2 and Theorem 3. Lemma 1. is well known.

Lemma 1. For each t in a dense subset D of the positive reals let $S(t)$ be a subset of E such that

- (i) $S(t) \subset S(u)$ if $t <' u$ and
- (ii) $\bigcup \{S(t): t \in D\} = E$.

For x in E take $f(x) = \inf \{t: x \in S(t)\}$. Then

$$\{x: f(x) <' u\} = \bigcup \{S(t): t \in D \text{ and } t <' u\}$$

and

$$\{x: f(x) \leq' u\} = \bigcap \{S(t): t \in D \text{ and } u <' t\}$$

for every real u .

Lemma 2. Let (E, k, k') be a bitopological space. For each t in a dense subset D of the positive reals let $S(t)$ be a subset of E such that

- (i) $i' S(t) = S(t)$
- (ii) $kS(t) \subset S(u)$ if $t <' u$ and
- (iii) $\bigcup \{S(t): t \in D\} = E$.

Then the function f from E to R defined by $f(x) = \inf \{t: x \in S(t)\}$ is continuous.

PROOF. For a real u the set $f^{-1}Ru = \{x: f(x) <' u\}$ is the union of i' -open sets and so is i' -open. Hence f is $k'-r'$ continuous.

Next, for a real u , the set $f^{-1}uR = \{x: u <' f(x)\}$ and so $cf^{-1}uR = \{x: f(x) \leq' u\} = \bigcap \{S(t): t \in D, u <' t\} = A$, say. Now $A \subset \bigcap \{kS(t): t \in D, u <' t\}$. Also

$$\bigcap \{kS(t): t \in D, u <' t\} \subset A$$

since $t \in D, u <' t$ imply there is v in D such that $u <' v <' t$ and so $kS(v) \subset S(t)$. Hence A is the intersection of k -closed sets and so is k -closed. Therefore $icA = cA$ and this implies f is $k-r$ continuous.

Theorem 3. Let $(E, <, k, k')$ be a bitopological space and let $A < B$. There is then a continuous function f from E to I such that $fA = 0$ and $fcB = 1$.

PROOF. Let D be the set of all numbers of the form $p2^{-q}$ where p and q are positive integers. Take $S(t) = E$ for t in D and $1 <' t$, take $S(1) = B$ and take $S(0)$ to be an i' -open set such that $A < S(0) < B$. For t in D and $0 <' t <' 1$ take t in the form $t = (2m+1)2^{-n}$ and choose, inductively on n , $S(t)$ to be an i' -open set such that $S(2m2^{-n}) < S(t) < S((2m+2)2^{-n})$. Such choice is possible since $<$ is a topogenous structure. Take $f(x) = \inf \{t: x \in S(t)\}$. Then f is continuous. Also $fA = 0$ and $fcB = 1$.

Corollary. $A < B$ implies there is a continuous function f from E to I such that $fkA = 0$ and $fk'cB = 1$.

Corollary. The bitopological space $(E, <, k, k)$ is completely regular.

Let S be a syntopogenous structure for E . Define $\mathcal{J}$ to be the family of all subsets T of E such that x in T implies $cT <' cx$ for some $<'$ in S . Then $\mathcal{J}$ is a topology for E . Similarly the family $\mathcal{J}'$, of all subsets T of E such that x in T implies $x <' T$ for some $<'$ in S , is also a topology for E .

Definition 7. We will say $(E, S, \mathcal{J}, \mathcal{J}')$ is the bitopological space of S , $\mathcal{J}$ is the left topology of S and $\mathcal{J}'$ is the right topology of S .

Given a syntopogenous structure S on E define a binary relation $<$ by $A < B$ iff $A <' B$ for some $<'$ in S . Then $<$ is a topogenous structure on E and the left and right topologies of $<$ coincide respectively with the left and right topologies of S . Hence a syntopogenous space (E, S) is completely regular. Also $A <' B$ for some $<'$ in S implies there is a continuous function f from E to I such that $fA = 0$ and $fB = 1$.

Definition 8. A bitopological space $(E, \mathcal{J}, \mathcal{J}')$ is said to be syntopogenizable (or topogenizable) iff there is a syntopogenous structure (or topogenous structure) on E whose bitopological space is $(E, \mathcal{J}, \mathcal{J}')$.

THAMPURAN [3] has proved that a completely regular bitopological space is quasiuniformizable. From a quasiuniformity $\mathcal{U}$ we can get a syntopogenous structure S — in the same way as a symmetric syntopogenous structure can be obtained from a uniformity — such that $\mathcal{U}$ and S have the same bitopological space.

It is clear that a bitopological space is topogenizable iff it is syntopogenizable. We now have the result:

Theorem 4. A bitopological space is completely regular iff it is topogenizable.

It is obvious that a subspace of a completely regular space is completely regular. Thampuran (2) has proved that a product of completely regular spaces is completely regular.

Definition 9. A bitopological space (M, k, k') is said to be regular iff

- (i) $A = kA$ and $y \in cA$ imply there are sets $X = iX$, $X' = i'X'$ such that $A \subset X'$ and $y \in X$ and $X \cap X' = \emptyset$ and
- (ii) $B = k'B$ and $x \in cB$ imply there are sets $Y = iY$, $Y' = i'Y'$ such that $x \in Y'$ and $B \subset Y$ and $Y \cap Y' = \emptyset$.

A completely regular space is evidently regular; hence a syntopogenous space is regular. It is clear that a subspace of a regular space is regular. A product of regular spaces has been shown to be regular by Thampuran (2).

Theorem 5. Let $<$ be a perfect topogenous structure on E . Then

- (i) $A \nless B$ iff $A \cap k'cB \neq \emptyset$ and
- (ii) $kA = \bigcup \{kx : x \in A\}$.

PROOF.

- (i) $A \nless B$ implies there is x in A such that $x \nless B$ and so x is in $k'cB$. Conversely, if there is x in A such that x is in $k'cB$ then $x \nless B$ and hence $A \nless B$.
- (ii) $x \in kA$ iff $A \nless cx$ and this holds iff there is y in A such that $y \nless cx$ or $x \in ky$.

We also have the following result for a perfect topogenous structure $<$ on E . If $<'$ is also a topogenous structure on E such that both $<$ and $<'$ have the same bitopological space (E, k, k') then $<$ is finer than $<'$, $A < cx$ implies $A <' cx$ and $x < cA$ implies $x <' cA$. But if $<'$ is also perfect then $< = <'$. These follow easily from Theorem 5., and from Corollary to Theorem 1.

References

- [1] Á. CSÁSZÁR, Foundations of general topology, *New York*, 1963.
- [2] D. V. THAMPURAN, Bitopological spaces and complete regularity (*to appear*).
- [3] D. V. THAMPURAN, Bitopological spaces and quasiuniformities (*to appear*).

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