The space of bounded maps into a Banach space

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Let D be a real B-space satisfying the following conditions: 1) D is strictly convex, 2) for every $v, v \in D^*$, $\|v\|=1$ the set $\{w, w \in D, \|w\| \leqq 1, v(w) = 1-\delta\}$ contains a u_1 for which

$\{w, w \in D, \|w-u_1\| \leqq r, v(w) = 1-\delta\} \subset \{w, w \in D, \|w\| \leqq 1, v(w) = 1-\delta\}$,

where $r/\delta \to \infty$ if $0<\delta<1$, $\delta \to 0$ and v is fixed, 3) D has no proper subspace isometrically isomorphic to D, 4) D is infinite dimensional. Let X be a realcompact space, i.e. completely regular and for every $p \in \beta X \setminus X$ there is an $f \in C(X)$ not extensible to p. $C^* = C^*(X, D)$ denotes the B-space of bounded continuous functions from X to D.

Theorem. *C^*, as a B-space, determines X and D. More accurately if for $C^*(X_i, D_i)$ (X_i, D_i belonging to the above classes) we have a linear isometry ψ of $C^*(X_1, D_1)$ onto $C^*(X_2, D_2)$, then there exist a homeomorphism $\varphi: X_2 \to X_1$ and a continuous map A from X_1 to the isometrical isomorphies of D_1 onto D_2 (taken in the strong operator topology) such that $(\psi f)(x_2) = A(\varphi(x_2)) \cdot f(\varphi(x_2))$.*

PROOF. 1. For later use we consider first the case $D=R$, and X is compact T_2 (condition 4 is not satisfied). We denote $C^*(X, R)$ by B.

$f \in B$ is an extremal point of the unit sphere if and only if $|f(x)|=1$ for every $x \in X$.

Let f_0 be an extremal point of the unit sphere. Let A be a continuous map from X to the isometrical isomorphies of R to itself.

After applying such an A to each element of B (like above, with φ=identity) we can suppose $f_0=1$. For $f \in B$ we consider $\|f-\lambda f_0\|$, where $\lambda \in R$. This is

$$\max(|\max f-\lambda|, |\min f-\lambda|).$$

Its minimum is attained for $\lambda = (\max f+\min f)/2$ and has value $(\max f-\min f)/2$. So we can determine $\max f$ and $\min f$.

Let $f_1, f_2 \in B$. We shall determine the convex hull $S(f_1, f_2)$ of the range $R(f_1, f_2)$ of the vector-valued function (f_1, f_2). This consists of those points $(x_1, x_2) \in R^2$ for which for every $(\lambda_1, \lambda_2) \in R^2$ we have $\min(\lambda_1 f_1+\lambda_2 f_2) \leqq \lambda_1 x_1+\lambda_2 x_2 \leqq \leqq \max(\lambda_1 f_1+\lambda_2 f_2)$.

For those elements $f \in B$ for which $\min f \geqq 0$ and $\max f \leqq 1$ we write $f_1 \sim f_2$ if $(x_1, x_2) \in S(f_1, f_2)$ implies that either $(x_1, x_2)=(0, 0)$ or $x_1>0$ and $x_2>0$. This condition is equivalent to the following: $(x_1, x_2) \in R(f_1, f_2)$ implies that either $(x_1, x_2)=(0, 0)$ or $x_1>0$ and $x_2>0$, i.e. $f_1^{-1}(0)=f_2^{-1}(0)$. Let the class of f by the equivalence relation $\sim$ be $\tilde{f}$. We say $\tilde{f}_1 < \tilde{f}_2$ if $g_1 \in \tilde{f}_1$, $g_2 \in \tilde{f}_2$, $(x_1, x_2) \in S(g_1, g_2)$, $x_1 > 0$ imply $x_2 > 0$. This means $f_1^{-1}(0) \supset f_2^{-1}(0)$.

Thus we obtained the lattice H (whose elements are the above $\tilde{f}$-s) dual to the lattice of the compact G_δ sets of X. By [3], p. 173 we obtain a homeomorphic copy of X, the points of which are the maximal ideals of H.

For $f \in B$ and $x \in X$ we determine $f(x)$. We can suppose $\max f - \min f \leqq 1$. Let $\min f \leqq c \leqq \max f$. Then $f = c - f_1 + f_2$ where $\min f_1 = 0$, $\min f_2 = 0$, $\max f_1 \leqq 1$, $\max f_2 \leqq 1$ and for every maximal ideal g of H one of $\tilde{f}_1$ and $\tilde{f}_2$ belongs to g (and this decomposition is unique). We say for those maximal ideals g that the value of f at the corresponding point is c, for which $\tilde{f}_1 \in g$ and $\tilde{f}_2 \in g$. If $\min f \leqq c \leqq \max f$ is not true, the value of f is not c in any point.

At the beginning we applied an operator A to each $f \in B$. This means that we determined X up to a homeomorphism and the functions $f \in B$ up to the application of some A. Thus for compact T_2 X_1 and X_2 and $D = R$ the linear isometries of $C^*(X_1, D)$ onto $C^*(X_2, D)$ are of the required form.

2. We turn to the proof of the theorem. Let C^* be given, at first we determine D and βX.

$f \in C^*$ is an extremal point of the unit sphere if and only if $\| f(x) \| = 1$ for every $x \in X$.

Let now $i: D \to C^*$ be any linear isometric embedding such that for every $d \in D$ id is an extremal point of the sphere with center in the origin and of radius $\|d\|$. Then for every $x \in X$ $\|(id)(x)\| = \|d\|$, thus $d \to (id)(x)$ is a linear isometry of D to a subspace of D, which by condition 3 has to be D. Thus the functions id can be carried over to the functions identically equal to d by the application of an operator A of the theorem (with φ = identity) cf. [1] Theorem 6. 1. Up to the application of such an A we can suppose $(id)(x) = d$ for every $x \in X$.

If C^* is isometrically isomorphic to $C^*(X', D')$ (X', D' from the above classes) then there is a linear isometric embedding $i': D' \to C^*$ having the same property as i, so D' can be embedded in D. Similarly vice versa, so D and D' are isometrically isomorphic.

For $f \in C^*$ let $S(f)$ denote the intersection of the closed spheres of centers d, d is any element of D, and radii $\|f - id\|$. $S(f)$ is a weakly closed convex set. We prove that it is the weakly closed convex hull of the range of f. That is, if a $u' \in D$ does not belong to some closed half-space $\{w', v(w') \leqq c\}$ containing the range of f, then it does not belong to $S(f)$ either.

We note that $\{w, \|w - u_1\| \leqq r, v(w) = 1 - \delta\} \subset \{w, \|w\| \leqq 1, v(w) = 1 - \delta\}$ implies $K = \{w, \|w - u_1\| \leqq r, v(w) \leqq 1 - \delta\} \subset \{w, \|w\| \leqq 1, v(w) \leqq 1 - \delta\}$ if $0 < \delta < 1$. This can be proved for two-dimensional subspaces by elementary geometry and the statement follows. There exist r', u_1' such that $v(u_1') = c$ and the range of f belongs to $\{w', v(w') \leqq c, \|w' - u_1'\| \leqq r'\} = K'$. We define u, u_0' by the condition that $\{u_0', K', u_1', u'\}$ could be obtained from $\{0, K, u_1, u\}$ by magnification and translation. If δ is sufficiently small u does not belong to the unit sphere since $v(u) = \big(v(u') - c\big)r/r' + 1 - \delta > 1$. So u' does not belong to the closed sphere of center u_0' and radius r', while K' belongs to it.

Let $0 \neq v \in D^*$. $v\big(S(f)\big)$ is the (closed, open or semi-open) interval determined by $\inf v(f)$, $\sup v(f)$. $vC^* = C^*(X)$. Like in 1 we determine βX and for every $f \in C^*$ $[v(f)]^\beta \in C^*(\beta X)$. For $f_i \in C^*$, $0 \leqq \inf v(f_i) \leqq \sup v(f_i) \leqq 1$, $i = 1, 2$, f_1 and f_2 are said to be equivalent if $v(f_1)\sim v(f_2)$. The equivalence class of f is denoted by $\tilde{f}^v$. These $\tilde{f}^v$-s are elements of the lattice H^v (ordered similarly as H). For $0\neq v_1, v_2$ the elements of H^{v_1} and H^{v_2} will be made to correspond so that the maximal ideals belonging to the same point $x\in\beta X$ correspond to each other. For linearly dependent v_1, v_2 this can be made easily. For linearly independent v_1, v_2 $\tilde{f}_1^{v_1}$ and $\tilde{f}_2^{v_2}$ will be made to correspond if for $g_1\in\tilde{f}_1^{v_1}$, $v_2(g_1)=0$, $g_2\in\tilde{f}_2^{v_2}$, $v_1(g_2)=0$ (such g_1, g_2 exist) $\tilde{g}_1^{v_1+v_2}=\tilde{g}_2^{v_1+v_2}$ (these are meaningful). The last equality is equivalent to $v_1(g_1)\sim v_2(g_2)$ (i.e. $\{[v_1(f_1)]^\beta\}^{-1}(0)=\{[v_2(f_2)]^\beta\}^{-1}(0)$). Thus we have determined βX and for every $f\in C^*$ and $0\neq v\in D^*$ $[v(f)]^\beta$.

We consider $D^{**}\supset D$ with the weak* topology. $f\in C^*$ can be extended to an $f^\beta:\beta X\to D^{**}$ continuous map, $\|f^\beta\|=\|f\|$ and for $v\in D^*$ $[v(f)]^\beta=v(f^\beta)$.

3. By condition 4 D contains $\{e_n\}_{n\in N}$ for which $\|e_n\|=1$ and for any $n, \lambda_0, \dots, \lambda_n$ $\|\lambda_0 e_0+\dots+\lambda_n e_n+e_{n+1}\|\geqq 1/2$. Let $R^+=\{x, x\in R, x\geqq 0\}$. We define $\varphi:R^+\to D$. φR^+ will belong to the unit sphere. For $x\in R^+$ $\varphi(x)=e_{[x]}\cdot(1-x+[x])+e_{[x+1]}\cdot(x-[x])$. φR^+ is closed in D. $\overline{\varphi R^+}$ denotes the closure of φR^+ in D^{**}. $(\overline{\varphi R^+}\setminus\varphi R^+)\cap D$ is closed in D. Let $x\in X$. If for $f\in C^*$ $f^\beta(\beta X)\subset\overline{\varphi R^+}$ and $f^\beta(x)\in\overline{\varphi R^+}\setminus\varphi R^+$ then for a closed-open neighbourhood U (in X) of x $f(U)\subset(\overline{\varphi R^+}\setminus\varphi R^+)\cap D$ so for a neighbourhood $V(=\overline{U})$ (in βX) of x $f^\beta(V)\subset\overline{\varphi R^+}\setminus\varphi R^+$. Let $x\in\beta X\setminus X$. Then there exists a continuous map $h:X\to R^+$ such that the extension $\bar{h}$ of h, $\bar{h}:\beta X\to R^+\cup\{\infty\}$ (one-point compactification) satisfies $\bar{h}(x)=\infty$. For $f=\varphi\circ h$ $f^\beta(\beta X)\subset\overline{\varphi R^+}$, $f^\beta(x)\in\in\overline{\varphi R^+}\setminus\varphi R^+$, but every neighbourhood of x contains an x' such that $f^\beta(x')\in\in\varphi R^+$.

Thus we have found the subspace X of βX. Thus we have determined X and found the value of every $f\in C^*$ at every $x\in X$ up to the application of an operator A of the theorem.

Remark. Let X be S-compact, where S is the unit sphere of D with the weak topology (i.e. let X be completely regular and for every $p\in\beta X\setminus X$ there is a continuous map $X\to S$ not extensible to p, cf. [4]). Let D have the above property 2. $C_w^*(X, D)$ denotes the B-space of the bounded weakly continuous functions from X to D. Let $C^*(X, D)\subset C^*\subset C_w^*(X, D)$ and let i map D into C^*, $(id)(x)=d$ for every $x\in X$. Then for the pair (i, C^*) a similar statement holds (with $A=$identity).

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Rerefences

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