

Krabbe's generalized functions as convolutions quotients

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In this paper applying the methods of [2] and [7] Krabbe's generalized functions (see [1]) will be embedded in a quotient ring which will be used for consideration of that.

A convolution algebra of locally integrable functions

Definition 1. Let $\mathcal{J} = \langle \alpha, \beta \rangle$ be an arbitrary (open, half-open or closed, bounded or unbounded) interval of the real line with endpoints α and β such that $0 \in \mathcal{J}$.

Let $\mathcal{L}(\mathcal{J})$ denote the vector space of locally Lebesgue-integrable complex-valued function on $\mathcal{J}$ over the complex number field $\mathcal{K}$.

For $f, g \in \mathcal{L}(\mathcal{J})$, let

$$(1) \quad (fg)(t) = \int_0^t f(t-\tau)g(\tau) d\tau \quad (t \in \mathcal{J}).$$

It is easily seen that $\mathcal{L}(\mathcal{J})$ with (1) is a commutative algebra over $\mathcal{K}$. Moreover it follows immediately from Titchmarsh's theorem (see [3]) that $f \in \mathcal{L}(\mathcal{J})$ ($f(t) \neq 0$ almost everywhere on $\mathcal{J}$) is a divisor of zero in $\mathcal{L}(\mathcal{J})$ if and only if there is a real number ϑ such that $0 < \vartheta < 1$ and $f(t) = 0$ almost everywhere on at least one of the sets $\vartheta\alpha < t \leq 0$ and $0 \leq t < \vartheta\beta$ which is not empty.

Definition 2. Let $\mathcal{T}(\mathcal{J})$ be the subset of $\mathcal{L}(\mathcal{J})$ to which all those functions w belong which are infinitely differentiable in $\mathcal{J}$ and satisfy $w^{(k)}(0) = 0$ for every integer $k \geq 0$.

Obviously $\mathcal{T}(\mathcal{J})$ is a subalgebra of $\mathcal{L}(\mathcal{J})$.

Convolution quotients

In theory of algebras it is known that every commutative algebra A over a field K with at least one element not a divisor of zero has a quotient ring Q which is a commutative ring with unit element. Furthermore K and A are embedded in Q as follows:

$$(2) \quad \alpha \stackrel{\text{def}}{=} \frac{ab}{b} \quad (\alpha \in K),$$

$$(3) \quad a \stackrel{\text{def}}{=} \frac{ab}{b} \quad (a \in A),$$

where $b \in A$ is not a divisor of zero in A .

Definition 3. Let $Q(\mathcal{J})$ be the quotient ring of $\mathcal{L}(\mathcal{J})$.

Remark 1. It is obvious that $Q([0, \infty))$ is the Mikusiński operator field (see [3]) and $Q([0, T])$ where $0 < T < \infty$ is the Mikusiński ring (see [4]).

It is easy to show that $Q(\langle \alpha, \beta \rangle)$ is the direct sum of $Q(\langle \alpha, 0 \rangle)$ and $Q([0, \beta])$ if $\alpha < 0 < \beta$, and $Q(\langle \alpha, 0 \rangle)$ and $Q([0, -\alpha])$ are isomorphic. In [5] it is proved that $Q([0, T])$ and $Q([0, T])$ are also isomorphic.

In this way the investigation of $Q(\mathcal{J})$ can be reduced to that of $Q([0, T])$ and $Q([0, \infty))$. In [6] an operational calculus belonging to $Q((-\infty, \infty))$ has been constructed in the preceding way.

Theorem 1. For each $q \in Q(\mathcal{J})$ there exist $w, \omega \in \mathcal{T}(\mathcal{J})$ such that $q = \frac{w}{\omega}$.

PROOF. Let τ be an element of $\mathcal{T}(\mathcal{J})$ which is not a divisor of zero in $\mathcal{L}(\mathcal{J})$. If $q = \frac{f}{g} \in Q(\mathcal{J})$ then $w = f\tau$ and $\omega = g\tau$ have the required properties.

Corollary 1.1. The quotient rings of $\mathcal{T}(\mathcal{J})$ and $\mathcal{L}(\mathcal{J})$ are isomorphic.

Definition 4. Let $\{\alpha\}$ denote the constant function of $\mathcal{L}(\mathcal{J})$ whose value is α .

Theorem 2. If $f \in \mathcal{L}(\mathcal{J})$ is locally absolutely continuous then for the convolution quotient $s = \frac{1}{\{1\}}$

$$(4) \quad sf = f' + f(0)$$

holds.

This formula is a generalization of the well-known formulas of Mikusiński (see [3] p. 349 and [4]), and can be proved quite similarly.

Generalized functions

Definition 5. Let $\mathcal{G}(\mathcal{J})$ be the set of all mappings F which map $\mathcal{T}(\mathcal{J})$ into $\mathcal{T}(\mathcal{J})$ such that

$$(5) \quad F(w\omega) = F(w)\omega$$

holds for all $w, \omega \in \mathcal{T}(\mathcal{J})$.

In $\mathcal{G}(\mathcal{J})$ let the usual equality, addition and composition of mappings be defined.

It will be shown that $\mathcal{G}(\mathcal{J})$ forms a commutative ring with unit element under the preceding operations.

Remark 2. By terms of [2] the elements of $\mathcal{G}(\mathcal{J})$ is maximal multiplier operators of $\mathcal{T}(\mathcal{J})$. Moreover, it is proved that the quotient algebra of a commutative algebra A (if it exists) is isomorphic to a subalgebra of the algebra of all maximal multipliers of A whose domain is without annihilator.

Theorem 3. Let ω_0 be a fixed element of $\mathcal{T}(\mathcal{J})$ which is not a divisor of zero in $\mathcal{T}(\mathcal{J})$. If $F \in \mathcal{G}(\mathcal{J})$, then

$$(6) \quad F(w) = \frac{F(\omega_0)}{\omega_0} w \quad (w \in \mathcal{T}(\mathcal{J})).$$

PROOF. Let $F \in \mathcal{G}(\mathcal{J})$, then for ω_0 and all $w \in \mathcal{T}(\mathcal{J})$ by (5) we have $F(w)\omega_0 = F(\omega_0)w$. Hence, because ω_0 is not a divisor of zero in $\mathcal{T}(\mathcal{J})$ the formula (6) follows.

Corollary 3.1. $\mathcal{G}(\mathcal{J})$ is a commutative ring with unit element.

Corollary 3.2. From (6) it follows that F is linear. Thus $\mathcal{G}(\mathcal{J})$ is the ring of Krabbe's generalized functions if $\mathcal{J}$ is open (see [1]).

Corollary 3.3. $F \in \mathcal{G}(\mathcal{J})$ has an inverse in $\mathcal{G}(\mathcal{J})$ if and only if the range of F is $\mathcal{T}(\mathcal{J})$, and then

$$(7) \quad F^{-1}(w) = \frac{\omega_0}{F(\omega_0)} w \quad (w \in \mathcal{T}(\mathcal{J}))$$

is the inverse of F in $\mathcal{G}(\mathcal{J})$.

PROOF. Suppose that $F \in \mathcal{G}(\mathcal{J})$ has an inverse in $\mathcal{G}(\mathcal{J})$, i.e., there exists a $G \in \mathcal{G}(\mathcal{J})$ such that $(FG)(w) = F(G(w)) = w$ for all $w \in \mathcal{T}(\mathcal{J})$. Thus $\text{rng } F = \mathcal{T}(\mathcal{J})$. On the other hand, then by (6) $F(w) = \frac{F(\omega_0)}{\omega_0} w$ and $G(w) = \frac{G(\omega_0)}{\omega_0} w$, and so $(FG)(w) = \frac{F(\omega_0)G(\omega_0)}{\omega_0^2} w$ ($w \in \mathcal{T}(\mathcal{J})$). Hence $F(\omega_0)G(\omega_0) = \omega_0^2$. Since ω_0 is not a divisor of zero in $\mathcal{T}(\mathcal{J})$ it follows that $F(\omega_0)$ is also not a divisor of zero in $\mathcal{T}(\mathcal{J})$ and $\frac{G(\omega_0)}{\omega_0} = \frac{F(\omega_0)}{\omega_0}$.

Conversely, if $\text{rng } F = \mathcal{T}(\mathcal{J})$ then there exists at least one $w_0 \in \mathcal{T}(\mathcal{J})$ such that $\omega_0 = F(w_0)$. Thus by (6) we have $\omega_0 = F(w_0) = \frac{F(\omega_0)}{\omega_0} w_0$ and so $F(\omega_0)w_0 = \omega_0^2$. Since ω_0 is not a divisor of zero in $\mathcal{T}(\mathcal{J})$, it follows that $F(\omega_0)$ is also not a divisor of zero in $\mathcal{T}(\mathcal{J})$. Now it is clear that $G(w) = \frac{\omega_0}{F(\omega_0)} w$ ($w \in \mathcal{T}(\mathcal{J})$) is the inverse of F in $\mathcal{G}(\mathcal{J})$.

Theorem 4.

$$(8) \quad F \stackrel{\text{def}}{=} \frac{F(\omega_0)}{\omega_0} \quad (F \in \mathcal{G}(\mathcal{J}))$$

is an embedding of $\mathcal{G}(\mathcal{I})$ in $Q(\mathcal{I})$ and after this embedding

$$(9) \quad \mathcal{G}(\mathcal{I}) \subset Q(\mathcal{I}), \mathcal{G}(\mathcal{I}) \neq Q(\mathcal{I})$$

holds.

We will show only that $\mathcal{G}(\mathcal{I}) \neq Q(\mathcal{I})$. Namely, for example $\frac{1}{\omega_0} \in Q(\mathcal{I})$ but $\frac{1}{\omega_0} \notin \mathcal{G}(\mathcal{I})$. Indeed let us suppose the contrary that $\frac{1}{\omega_0} \in \mathcal{G}(\mathcal{I})$. Then for $F = \frac{1}{\omega_0}$ by (8) $\frac{1}{\omega_0} = \frac{F(\omega_0)}{\omega_0}$ follows, and so $F(\omega_0) = 1$ is a contradiction.

Remark 3. In this sense arbitrary convolution quotient q is identifiable with the mapping

$$(10) \quad F_q(w) = qw \quad (w \in \{w \in \mathcal{T}(\mathcal{I}) : qw \in \mathcal{T}(\mathcal{I})\})$$

and so the convolution quotient q is a linear operator which maps an ideal of $\mathcal{T}(\mathcal{I})$ (with at least one element not a divisor of zero in $\mathcal{T}(\mathcal{I})$) into $\mathcal{T}(\mathcal{I})$ satisfying the equation (5) such that it has no proper extension in $\mathcal{T}(\mathcal{I})$.

Now the following statement is quite obvious:

Theorem 5. A convolution quotient q is an element of $\mathcal{G}(\mathcal{I})$ if and only if $qw \in \mathcal{T}(\mathcal{I})$ for all $w \in \mathcal{T}(\mathcal{I})$.

Corollary 5.1. $K \subset \mathcal{G}(\mathcal{I})$.

Corollary 5.2. $\mathcal{L}(\mathcal{I}) \subset \mathcal{G}(\mathcal{I})$.

Corollary 5.3. $s \in \mathcal{G}(\mathcal{I})$ and thus the formula (4) holds also in $\mathcal{G}(\mathcal{I})$.

Definition 6. Let $\{\mathcal{L}(\mathcal{I}), s\}$ denote the subring of $Q(\mathcal{I})$ generated by $\mathcal{L}(\mathcal{I})$ and s .

Corollary 5.4. $\{\mathcal{L}(\mathcal{I}), s\} \subset \mathcal{G}(\mathcal{I})$.

Problem. Is the equality $\mathcal{G}(\mathcal{I}) = \{\mathcal{L}(\mathcal{I}), s\}$ true?

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