

Remark on the paper Á. Szász: Krabbe's generalized functions as convolution quotients

By E. GESZTELYI (Debrecen)

Let $\mathcal{J} = \langle \alpha, \beta \rangle$ be an arbitrary bounded or unbounded interval of the real line such that $0 \in \mathcal{J}$. Let $\mathcal{L}(\mathcal{J})$ be the convolution algebra of locally Lebesgue-integrable complex-valued functions on $\mathcal{J}$. Thus the multiplication in $\mathcal{L}(\mathcal{J})$ is the convolution

$$(fg)(t) = \int_0^t f(t-\tau)g(\tau) d\tau \quad (f, g \in \mathcal{L}(\mathcal{J}); t \in \mathcal{J}),$$

while the addition and the multiplication by scalars are usually defined. Since $\mathcal{L}(\mathcal{J})$ contains elements which are not divisors of zero, $\mathcal{L}(\mathcal{J})$ has a quotient ring $Q(\mathcal{J})$.

Let $\mathcal{T}(\mathcal{J})$ be the set of functions $\varphi \in \mathcal{L}(\mathcal{J})$ which are infinitely differentiable on $\mathcal{J}$ and such that $\varphi^{(k)}(0) = 0$ for every integer $k \geq 0$.

A mapping $F: \mathcal{T}(\mathcal{J}) \rightarrow \mathcal{T}(\mathcal{J})$ is called a multiplier operator of $\mathcal{T}(\mathcal{J})$ (see [1]) if

$$F(\varphi\psi) = F(\varphi)\psi$$

holds for all $\varphi, \psi \in \mathcal{T}(\mathcal{J})$. The set of all multiplier operators of $\mathcal{T}(\mathcal{J})$ will be denoted by $\mathcal{G}(\mathcal{J})$.

Á. SZÁSZ has shown ([2]) (after the embedding of $\mathcal{G}(\mathcal{J})$ in $Q(\mathcal{J})$) that

$$(1) \quad \{\mathcal{L}(\mathcal{J}), s\} \subseteq \mathcal{G}(\mathcal{J})$$

where $\{\mathcal{L}(\mathcal{J}), s\}$ is the subring of $Q(\mathcal{J})$ generated by $\mathcal{L}(\mathcal{J})$ and $s = \frac{1}{\{1\}}$. He has the following question stated: *Is the equality*

$$\mathcal{G}(\mathcal{J}) = \{\mathcal{L}(\mathcal{J}), s\}$$

true? In the present note we give a negative answer for this question.

Obviously, it is sufficient to verify only in a special case the relation

$$(2) \quad \{\mathcal{L}(\mathcal{J}), s\} \subset \mathcal{G}(\mathcal{J}).$$

Next we shall suppose that $\mathcal{J} = [0, \infty)$ and in this case $Q(\mathcal{J})$ is the field of Mikusiński operators ([3]).

It should be remarked that for every element a of $\{\mathcal{L}(\mathcal{I}), s\}$ there exists a positive integer k and a continuous function f such that

$$(3) \quad a = s^k f \quad (f \in \mathcal{C}).$$

It is clear that the operator

$$(4) \quad b = \sum_{v=0}^{\infty} s^v e^{-vs}$$

cannot be written in the form (3). Consequently $b \notin \{\mathcal{L}(\mathcal{I}), s\}$. However $b \in \mathcal{G}(\mathcal{I})$. Let, namely, φ be an arbitrary element of $\mathcal{T}(\mathcal{I})$. We extend φ for $(-\infty, 0)$ as follows: $\varphi(t) = 0$ if $t < 0$. Then we obtain

$$b\varphi = \sum_{v=0}^{\infty} s^v e^{-vs} \varphi = \left\{ \sum_{v=0}^{\infty} \varphi^{(v)}(t-v) \right\} \in \mathcal{T}(\mathcal{I}).$$

Thus $b \in \mathcal{G}(\mathcal{I})$, since, by Theorem 5. of [2], b is an element of $\mathcal{G}(\mathcal{I})$ iff $b\varphi \in \mathcal{T}(\mathcal{I})$ for all $\varphi \in \mathcal{T}(\mathcal{I})$.

References

- [1] L. MÁTÉ, Multiplier operators and quotient algebra, *Bull. Acad. Polonaise*, **8**, (1965), 523—526.
- [2] Á. SZÁSZ, Kabbe's generalized functions as convolution quotients. *In this journal*
- [3] J. MIKUSIŃSKI, Operational calculus, *New York*, 1959.

(Received November 8, 1971.)