Regularity for bitopological spaces

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A topological space X is said to be completely regular iff A is a closed subset of X and x is a point not in A imply there is a continuous function f from X to the closed unit interval $[0, 1]$ such that $f(x)$ is 0 and f is 1 on A. WEIL (4) introduced uniform spaces and showed that a topological space is completely regular iff it is uniformizable. THAMPURAN [1, 2] has proved similar results for regular topological spaces and completely regular bitopological spaces: a topological space is regular iff its topology is that of a regularity (a structure similar to a uniformity) and a bitopological space is completely regular iff it is quasiuniformizable. The object of this paper is to show that a bitopological space is regular iff it is the space of a quasiregularity.

Let X be a set. If A is a subset of X and U is a subset of $X\times X$, take $AU=\{y:(x, y)\in U$ for some x in $A\}$ and $UA=\{y:(y, x)\in U$ for some x in $A\}$; if A contains only one point x then we will write xU and Ux for AU and UA. By xUU is meant $(xU)U$ and $UUx=U(Ux)$. For a subset A of X take $cA = X-A$.

Let X be a set and $\mathscr{U}$ a family of subsets of $X\times X$ such that for x in X

1. (x, x) is in each member of $\mathscr{U}$
2. U in $\mathscr{U}$ implies there are V, W in $\mathscr{U}$ such that $xVV\subset xU$ and $WWx\subset Ux$
3. U, V in $\mathscr{U}$ and x in X imply there are W, W' in $\mathscr{U}$ such that $xW \subset xU\cap xV$ and $W'x \subset Ux\cap Vx$ and
4. $U \subset V \subset X\times X$ and U in $\mathscr{U}$ imply V is in $\mathscr{U}$.

Definition 1. $\mathscr{U}$ as defined above is said to be a quasiregularity for X and $(X, \mathscr{U})$ is said to be a quasiregular space.

Definition 2. Let k, k' be two Kuratowski closure functions for X. Then (X, k, k') is said to be a bitopological space. A statement concerning the topological space (X, k) is said to be a k-statement and a k'-statement will have a similar meaning.

Take $i=ckc$, $i'=ck'c$.

Let $(X, \mathscr{U})$ be a quasiregular space. Let $\mathscr{I}$ be the family of all subsets T of X such that x in T implies $Ux\subset T$ for some U in $\mathscr{U}$; it is obvious that $\mathscr{I}$ is a topology for X. The family $\mathscr{I}'$, of all subsets T of X such that x in T implies $xU\subset T$ for some U in $\mathscr{U}$, is also a topology for X.

Definition 3. $\mathscr{I}$, $\mathscr{I}'$ as defined above are said to be the left and right topologies respectively of $\mathscr{U}$. Denote by k, k' the closure functions for $\mathscr{I}$, $\mathscr{I}'$; then (X, k, k') is said to be the bitopological space of $\mathscr{U}$.

Theorem 1. *Let $(X, \mathscr{U}, k, k')$ be the bitopological space of a quasiregularity $\mathscr{U}$ and let A be a subset of X. Then $iA = \{x : Ux \subset A$ for some U in $\mathscr{U}\}$ and $i'A = \{x : xU \subset A$ for some U in $\mathscr{U}\}$.*

Proof. Let $B = \{x : Ux \subset A$ for some U in $\mathscr{U}\}$. Obviously $iA \subset B \subset A$ and so $iB = B$ implies $iA = B$. Let x be a point of B. Then $Ux \subset A$ for some U in $\mathscr{U}$. There is now V in $\mathscr{U}$ such that $VVx \subset Ux$. Let $y \in Vx$. Then $Vy \subset VVx \subset Ux \subset A$ and so $y \in B$. Hence $Vx \subset B$ which implies $iB = B$. Then proof for $i'A$ is similar.

Corollary. Ux is a k-neighborhood and xU is a k'-neighborhood of x for each U in $\mathscr{U}$.

Definition 4. A bitopological space (X, k, k') is said to be regular iff

1. A is a k-closed subset of X and y is in cA imply A has a k'-neighborhood and y has a k-neighborhood which are disjoint and
2. B is a k'-closed subset of X and x is in cB imply B has a k-neighborhood and x has a k'-neighborhood which are disjoint.

Theorem 2. *Let $(X, \mathscr{U})$ be a quasiregular space. Then its bitopological space $(X, \mathscr{U}, k, k')$ is regular.*

Proof. Let A be a k-closed subset of X and let y be in cA. Then there is U in $\mathscr{U}$ such that $Uy \subset cA$. Hence there is V in $\mathscr{U}$ such that $VVy \subset Uy$. This means AV is a k'-neighborhood of A, Vy is a k-neighborhood of y and AV and Vy are disjoint. The other part can be proved similarly.

Definition 5. Let X be a set. A set-valued set-function h mapping the power set, of X, to itself is said to be a neighborhood function for X iff

1. $h\emptyset = \emptyset$
2. $A \subset hA$ for $A \subset X$ and
3. $hA \subset hB$ if $A \subset B \subset X$.

The ordered pair (X, h) is said to be a neighborhood space. A subset A of X is said to be a h-neighborhood or neighborhood of a point x iff $x \in chcA$.

Definition 6. Let (X, h), (Y, p) be two neighborhood spaces and f a function from X to Y. We will say f is continuous at a point x of X iff the inverse under f of each neighborhood of $f(x)$ is a neighborhood of x. The function f is said to be (h, p)-continuous iff it is continuous at each point of X.

Definition 7. Let h, h' be two neighborhood functions for a set X. Then the ordered triple (X, h, h') is said to be a bineighborhood space. A function f from a bineighborhood space (X, h, h') to a bineighborhood space (Y, p, p') is said to be continuous iff f is both (h, p) and (h', p')-continuous.

Let N denote the set of points $1, 1/2, 1/3, \ldots, 0$. Let u, v denote points of N. Define a distance function e for N as follows.

$$e(u, v) = \begin{cases} v - u & \text{if } u < w < v \text{ for some } w \text{ in } N \\ 0 & \text{otherwise.} \end{cases}$$

For $r>0$ let $V(r)=\{(u, v):e(u, v)<r\}$. Define the neighborhood functions n, n' for N as follows. For a subset M of N let nM be the set of all points u such that $V(r)u$ intersects M for each $r>0$ and let $n'M$ be the set of all points u such that $uV(r)$ intersects M for all $r>0$.

THAMPURAN (3) has proved the following results: Let (X, k, k') be a regular bitopological space. Then

1. B is a k'-closed subset of X and x is in cB imply there is a continuous function f from (X, k, k') to (N, n, n') such that $f(x)$ is 0 and f is 1 on B and
2. A is a k-closed subset of X and y is in cA imply there is a continuous function g from (X, k, k') to (N, n', n) such that $g(y)=0$ and g is 1 on A.

Theorem 3. *A bitopological space (X, k, k') is regular iff it is the bitopological space of a quasiregularity $\mathscr{U}$ for X.*

PROOF. Let the space be regular. If B is a k'-closed subset of X and x a point of cB, then there is a continuous function f from (X, k, k') to (N, n, n') such that $f(x)$ is 0 and f is 1 on B; for y, z in X write $d(y, z)=e(f(y), f(z))$. There is such a d for each k'-closed set B and each x in cB; let D be the family of all such d. For each k-closed set A and each x in cA there is a continuous function f' from (X, k, k') to (N, n', n) such that $f'(x)$ is 0 and f' is 1 on A; for y, z in X write $d'(y, z)=e(f(z), f(y))$. For each k-closed set A and each x in cA there is such a d'; let D' be the family of all such d'. Take $E = D\cup D'$.

For d in E and $r>0$ take $V(d, r)=\{(y, z):d(y, z)<r,\ y,\ z \text{ in } X\}$. For d in D, consider a $U=V(d, r)$ and let x be in X. Now xU is a k'-neighborhood of x. Take $cB=i'(xU)$. Then there is a continuous function g from (X, k, k') to (N, n, n') such that $g(x)=0$ and g is 1 on B; for y, z in X take $b(y, z)=e(g(y), g(z))$. Let $V=V(b, 1/8)$. Let s be in xV. Then $b(x, s)<1/8$ and so $g(s)<1/8$. If t is in sV then $g(t)<1/4$ and so t is in cB. Hence $xVV\subset xU$. Next, Ux is a k-neighborhood of x and so there is a continuous function g' from (X, k, k') to (N, n', n) such that $g'(x)$ is 0 and g' is 1 on A where $cA=i(Ux)$; for y, z in X take $b'(y, z)=e(g'(z), g'(y))$. Let $V'=V'(b, 1/8)$. Then $V'V'x\subset Ux$.

Similarly we can prove that x in X and $U=V(d, r)$ for d in D' imply there are $W=(p, 1/8)$, $W'=V(q, 1/8)$ for some p, q in E such that $xWW\subset xU$ and $W'W'x\subset Ux$.

Let $\mathscr{U}$ be the family of all subsets U of $X\times X$ such that U contains the intersection of a finite number of the sets $V(d, r)$ for d in E and $r>0$. It is obvious that $\mathscr{U}$ is a quasiregularity for X.

Finally, let $\mathscr{S}$, $\mathscr{S}'$ be the left and right topologies respectively of $\mathscr{U}$. We know that $x\in X$ and $V=V(d, r)$, for d in E and $r>0$, imply that xV is a k'-neighborhood and Vx is a k-neighborhood of x. Hence x is in X and U is in $\mathscr{U}$ imply xU and Ux are k' and k-neighborhoods respectively of x. Let $S\in\mathscr{S}$ and $x\in S$; then $Ux\subset S$ for some U in $\mathscr{U}$ and so S is k-open. If T is k-open and $x\in T$ then there is d in D' such that $Ux\subset T$ where $U=V(d, 1/4)$ and so $T\in\mathscr{S}$. Similarly, a subset S of X is k'-open iff $S\in\mathscr{S}'$. Hence (X, k, k') is the bitopological space of $\mathscr{U}$.

It has already been proved that the bitopological space of a quasiregularity is regular. This completes the proof.

It may be noted that the $\mathscr{U}$ obtained in the proof of Theorem 3 is much more than a quasiregularity. The intersection of two members of $\mathscr{U}$ is a member of $\mathscr{U}$ and U in $\mathscr{U}$ and x in X imply there is V in $\mathscr{U}$ such that $xVV\subset xU$ and $VVx\subset Ux$.

References

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