

Regular congruence on a simple semigroup with a minimal right ideal

By BRUCE W. MIELKE (Providence, R. J.)

In the first part of this paper, two characterizations of a group congruence on a simple semigroup with a minimal right ideal are given (1. 3), (1. 9). The conditions given in (1. 3) generalize Schwarz's conditions for a group congruence on a completely simple semigroup (1. 4).

In the second section, it is shown that any regular¹ congruence on a right simple semigroup is characterized as the intersection of a group congruence and a band congruence (2. 5). It is also shown that every simple semigroup with a minimal right ideal has a minimum completely simple congruence (2. 7).

Since all regular congruences on simple semigroups with a right ideal are completely simple congruences, it follows that the lattice of regular congruences for such semigroups is isomorphic to the lattice of congruences on a completely simple semigroup (2. 8). This lattice is described by KAPP and SCHNEIDER [3].

The basic properties of simple semigroups with a minimal right ideal may be found in section 8. 2 [1]. The terminology and notation will be that of CLIFFORD and PRESTON [1].

1. Group congruences

It is clear that a right simple semigroup is simple with a minimal right ideal. For the convenience of the reader, we will state without proofs TEISSIER's characterization of a group congruence [6] on such a semigroup.

Recall the following:

(1. 1) *Definition.* ([1], 55, Vol. II.) A subset U of a semigroup S is said to be *left [right] unitary* in S if $u \in U$ and $ux \in U$ [$xu \in U$], for $x \in S$, together imply that $x \in U$. A subset U which is both right and left unitary in S is said to be *unitary* in S .

(1. 2) *Theorem.* ([6]). A right simple semigroup, S , without idempotent elements has a group congruence ρ if and only if S contains a subsemigroup, E , which is unitary in S and satisfies $Eae \subseteq aE$ for all $ae \in SE$. Moreover, E is the kernel of ρ and aqb for all $a, b \in S$, if and only if $aE = bE$.

¹) Let P be an abstract property of semigroups. Then a congruence ρ of the semigroup S has the property P if S/ρ has this property. E. g. P can be: to be group, regular semigroup.

One may use techniques similar to those used by Teissier to generalize (1. 2).

(1. 3) **Theorem.** *Let S be a simple semigroup with a minimal right ideal. Then S has a group congruence, ϱ , if and only if S contains a unitary subsemigroup, E , such that for all $x, y \in S$, $xEy \subseteq ExyE$. Moreover, E is the kernel of ϱ , and $(a, b) \in \varrho$ if and only if $EaE = EbE$.*

We note here that in (1. 3), unlike (1. 2), S is not required to be idempotent free, and that a simple semigroup with a minimal right ideal is completely simple if and only if it contains an idempotent ([1] Theorem 8. 14). Thus as a corollary to (1. 3), we have:

(1. 4) **Corollary** ([5] (4) Theorem.) *Let S be a completely simple semigroup. Then S has a group congruence if and only if S contains a simple semigroup, E , which contains all of the idempotents of S , and for which there exists at least one H -class, H_a of S such that $E \cap H_a$ is a normal subgroup of H_a . In this case, E is the kernel of the congruence, and the congruence classes are of the form ExE for $x \in S$.*

We now give another characterization of a group congruence on a simple semigroup with a minimal right ideal.

(1. 5) **Lemma.** ([1] Lemma 8. 13.) *Let S be a simple semigroup with a minimal right ideal. Then S is the disjoint union of its minimal right ideals; sS is the minimal right ideal containing s ; every minimal right ideal is a right simple semigroup.*

(1. 6) **Definition.** Let E be a subset of a semigroup T . We say E is a *right cross-section* if $E \cap R_a \neq \square$, for all $a \in T$. $\square$ denoting the empty set.

(1. 7) **Note.** It is easily checked that if E is a class of a congruence ϱ on a semigroup T , then if $x, y \in EaE$ for any $a \in T$, $(x, y) \in \varrho$.

(1. 8) **Lemma.** *Let ϱ be a congruence on S . Let E be a ϱ -class which is also a unitary subsemigroup of S and satisfies:*

i) *E is a right cross-section.*

ii) *For any $a \in S$, there exists $e \in E$ such that $a = ae$.*

Then for any ϱ -class, U , there exists $A \subseteq S$ such that $U = \bigcup_{x \in A} ExE$.

PROOF. Let $s \in U$, then by i), $E \cap sS \neq \square$. By (1. 5), sS is the minimal right ideal of S containing s , and hence if $e \in E \cap sS$, $eS = sS$. But then there exists $s_1 \in S$, such that $s = es_1$. By ii), there exists $e_1 \in E$, such that $s_1 = s_1e_1$. We combine these equations to get $s = es_1e_1 \in Es_1E$, so that every element of U is contained in a set ExE for some $x \in S$, and there exists a subset A of S , such that $U \subseteq \bigcup_{x \in A} ExE$. But from (1. 7), it is clear that if $(ExE) \cap U \neq \square$ for any $x \in S$, then $ExE \subseteq U$. It now follows that there exists a subset A of S such that $U = \bigcup_{x \in A} ExE$.

The second characterization follows:

(1. 9) **Theorem.** *Let S be a simple semigroup with a minimal right ideal. Then if S has a group congruence, ϱ , E , the kernel of ϱ is a unitary subsemigroup such that:*

i) *E is a right cross-section.*

ii) *For $a \in S$, $a = ae$ for some $e \in E$.*

iii) *If $xEy \cap E \neq \square$, for $x, y \in S$, then $xEy \subseteq E$.*

Conversely, if E is a unitary subsemigroup of S satisfying i)—iii), then there exists a group congruence, ϱ , with E as its kernel.

PROOF. Suppose that S has a group congruence, ϱ . Then by (1.3), the kernel of ϱ , E , is a unitary subsemigroup of S , such that for all $x, y \in S$, $xEy \subseteq ExyE$. The $\mathbf{R}$ -classes of S are of the form aS where $a \in S$ (1.5), so that E is a right cross-section if for every $a \in S$, $E \cap aS \neq \emptyset$. But the collection of sets EaE forms a group, (1.3) and each element EaE has an inverse EbE which satisfies $(EaE)(EbE) = EabE = E$. It is easily shown that this implies $ab \in E$, but $ab \in aS$, hence E is a right cross-section. If $aEb \cap E \neq \emptyset$, we apply (1.3) to get $EabE \cap E \supseteq aEb \cap E \neq \emptyset$. But then by (1.7), it follows that $EabE = E$. Thus $aEb \subseteq EabE = E$. Therefore i) and iii) hold. We now show that ii) is valid. Let $a \in S$. We know that aS is the minimal right ideal containing a (1.5) and therefore $a = as$ for some $s \in S$. By (1.3), the ϱ -classes of S are of the form ExE , and we have $(EaE)(EsE) = EasE = EaE$. Since S/ϱ is a group, EsE must be the group identity, E . Thus $s \in E$ for E is unitary, and we have ii).

Conversely, suppose that S has a unitary subsemigroup, E , satisfying i)—iii). TEISSIER ([7], 2) has shown that condition iii) is a necessary and sufficient condition for the existence of a congruence, ϱ , for which E is a ϱ -class. For every ϱ -class, U , of S there exists a subset A of S , such that $U = \bigcup_{x \in A} ExE$, by (1.8). Clearly $E^2 \subseteq E$, so that E is a right identity for S/ϱ . To show S/ϱ is a group, we need only show that every ϱ -class, U , has a right inverse. Let $a \in S$ for which $EaE \subseteq U$. By i), $E \cap aS \neq \emptyset$, and there is an $s \in S$ such that $as \in E$. Let U' be the ϱ -class containing EsE , then $UU' \supseteq (EaE)(EsE)$. By ii), there exists $e \in E$ such that $a = ae$. Then we have $aes = as \in aEs \cap E$, and by iii), $aEs \subseteq E$. Thus $UU' = E$, and U has a right inverse. Therefore S/ϱ has a right identity and every element of S/ϱ has a right inverse, hence, S/ϱ is a group.

The following is a simple example of a non-trivial group homomorphism on a simple semigroup with more than one minimal right ideal and without idempotents. For a more complex example, see [4].

(1.10) *Example.* Let $S = A \times B$ the direct product of a nontrivial left zero semigroup, A , and a Baer—Levi semigroup, B , of type (p, q) with $p > q$ ([1] § 8.1). Since B is right simple idempotent free, S is simple, and $\lambda \times B$ is a minimal right ideal for each $\lambda \in A$. Clearly S has no idempotent. Define the map $\alpha: S \rightarrow B$ as follows: $(\lambda, a)\alpha = a$, for all $(\lambda, a) \in S$. It is easily shown that α is a homomorphism of S onto B . One can use (1.2) to show that there is a non-trivial homomorphism, δ , of B onto a group [4]. Clearly $\alpha\delta$ is a non-trivial homomorphism of S onto a group.

The following lemma is easily proven.

(1.11) **Lemma.** Let S be a semigroup with subsemigroup T , and let ϱ be a congruence on S . Then $\varrho/T = \{(x, y): x, y \in T, (x, y) \in \varrho\}$ is a congruence on T , and if $a(\varrho/T)$ is the ϱ/T -class of $a \in T$, then $a(\varrho/T) = T \cap a\varrho$, where $a\varrho$ is the ϱ -class of a .

(1.12) *Proposition.* Let S be a simple semigroup with a minimal right ideal. If τ is a group congruence on S , then for any $a \in S$, $aS/(\tau/aS)$ is isomorphic to S/τ .

PROOF. Let E be the kernel of τ . It is easily checked that $ExE \cap aS \neq \emptyset$ for any $x \in S$, since by (1.9), E is a right cross-section. From this it follows that for

every $x \in S$, there a $y \in aS$, for which $ExE = EyE$. Thus if for every $x \in aS$ we define $(ExE)\theta = ExE \cap aS$, one can easily check that θ is an isomorphism of S/τ onto $(aS)/(\tau/aS)$.

2. The lattice of regular congruences on simple semigroups with a minimal right ideal

We recall that in general, a semigroup need not have a minimum group congruence. However, we now show that every right simple semigroup has a minimum group congruence.

(2.1) *Proposition.* Let S be a right simple semigroup, then S has a minimum group congruence.

PROOF. One can easily show that S has a minimum cancellative congruence, μ , by ([1], Theorem 1.7). S/μ is right simple and left cancellative, hence a right group ([1] 1.1). Thus S/μ has an idempotent, but it is also right cancellative, therefore S/μ is a group ([1] vol. II, 85, ex. 5). If σ is any group congruence, then σ is a cancellative congruence, and $\mu \subseteq \sigma$. Thus μ is the minimum group congruence on S .

We recall that the homomorphic image of a simple semigroup with a minimal right ideal is a simple semigroup with a minimal right ideal. Hence, if the homomorphic image of such a semigroup has an idempotent, it is a completely simple semigroup ([1] Theorem 8.14). Thus every regular congruence on a simple semigroup with a minimal right ideal is a completely simple congruence.

We will now find a minimum completely simple congruence.

(2.2) *Notation.* Let S be a simple semigroup with a minimal right ideal. Then for every $a \in S$, aS is a right simple semigroup (1.5), so that aS has a minimum group congruence, which we denote by γ_a , and a minimum band congruence, which we denote by β_a .

We now quote a theorem of HOWIE and LALLEMENT [2] which will form the basis for the main result.

(2.3) **Theorem.** ([2] Theorem 4.1.) *Let S be a regular semigroup. If τ is a group congruence on S , and if σ is a band congruence on S , then $S/(\tau \cap \sigma)$ is a band of groups whose idempotents form a unitary subsemigroup. Conversely, if ϱ is a congruence on S and S/ϱ is a band of groups whose idempotents form a unitary subsemigroup, then $\varrho = \tau \cap \sigma$ where τ is a group congruence on S and σ is a band congruence on S . Moreover, τ and σ are uniquely determined by ϱ .*

(2.4) **Lemma.** *Let S be a right simple semigroup. If τ is a group congruence on S , and if σ is a band congruence on S , then $S/(\tau \cap \sigma)$ is regular.*

PROOF. If E is the kernel of τ , and $x \in E$, then $(x, x^2) \in \tau$. Clearly $(x, x^2) \in \sigma$, therefore $(x, x^2) \in \tau \cap \sigma$. Thus $S/(\tau \cap \sigma)$ is right simple with an idempotent, hence it is regular.

Next we generalize (2.3) to right simple semigroups.

(2.5) **Theorem.** *Let S be a right simple semigroup. If τ is a group congruence on S , and if σ is a band congruence on S , then $S/(\tau \cap \sigma)$ is regular. Moreover, if σ is a regular congruence on S , then $\varrho = \tau \cap \sigma$ where τ is a group congruence on S and σ is a band congruence on S . In this case, τ and σ are uniquely determined by ϱ .*

PROOF. $\tau \cap \sigma$ is a regular congruence by (2. 4).

Let ϱ be a regular congruence on S . Then S/ϱ is a right group ([1] Theorem 1. 27), hence isomorphic to $G \times E$ where G is a group and E is a right zero semigroup ([1] Theorem 1. 27). One easily checks that the idempotents of $G \times E$ are of the form (e, μ) where e is the identity of G and μ is an arbitrary element of E . It is clear from this that the idempotents of $G \times E$, and hence of S/ϱ , form a unitary subsemigroup. Since S/ϱ is a regular semigroup, we may now apply (2. 3) to $\Delta_{S/\varrho}$, the identity relation on S/ϱ , to get $\Delta_{S/\varrho} = \tau' \cap \sigma'$ where $\tau'[\sigma']$ is a group [band] congruence on S/ϱ . By ([1], Theorem 1. 5), there exists $\tau[\sigma]$ a group [band] congruence on S containing ϱ such that $\tau' = \tau/\varrho$ [$\sigma' = \sigma/\varrho$]. Then $(\tau/\varrho \cap (\sigma/\varrho)) = \Delta_{S/\varrho}$ and thus $\varrho = \tau \cap \sigma$. ϱ is regular, therefore τ' and σ' are uniquely determined by (2. 4), and hence τ and σ are uniquely determined by ϱ .

(2. 6) **Lemma.** *Let S be a simple semigroup with a minimal right ideal, and π be the congruence generated by the relation $\alpha = \bigcup_{x \in A} (\gamma_x \cap \beta_x)$. Then S/π is a completely simple semigroup.*

PROOF. Clearly S/π is simple with a minimal right ideal. Let

$$\pi/aS = \{(x, y) : x, y \in S, (x, y) \in \pi\}$$

for $a \in S$. By (1. 11), π/aS is a congruence on aS , and $\gamma_a \cap \beta_a \subseteq \pi/aS$. Then $aS/(\pi/aS)$ is regular, since by ([1] Corollary 1. 62), it is the homomorphic image of $aS/(\gamma_a \cap \beta_a)$ which is regular (2. 3). Let $e(\pi/aS)$ be an idempotent element of $aS/(\pi/aS)$, then $e^2 \in e^2(\pi/aS) = [e(\pi/aS)]^2 = e(\pi/aS)$. But we have noted in (1. 11) that $e(\pi/aS) = e\pi \cap aS$, therefore $e^2 \in e\pi$. Since $e^2 \in e^2\pi = [e\pi]^2$ and π is an equivalence relation, we have $[e\pi]^2 = e\pi$, so that $e\pi$ is an idempotent element of S/π . Thus S/π is simple with a minimal right ideal, and has an idempotent, therefore it is completely simple by ([1], Theorem 8. 14).

(2. 7) **Theorem.** *Let S be a simple semigroup with a minimal right ideal. Then π , as in (2. 6), is the minimum completely simple congruence on S .*

PROOF. Let ϱ be any completely simple congruence on S . For every $a \in S$, it is easily shown that the collection of all ϱ -classes of S which have non-trivial intersection with aS is an $\mathbf{R}$ -class of S/ϱ . So there exists $e \in aS$ for which $e\varrho = (e\varrho)^2$. But then, $e\varrho \cap aS = (e\varrho)^2 \cap aS$ and by (1. 11), we have ϱ/aS is a regular congruence on aS . We recall that all regular congruences on right simple semigroups can be written as the intersection of a group congruence and a band congruence (2. 5). Therefore $\varrho/aS = \tau \cap \sigma$, where τ is a group congruence on aS and σ is a band congruence on aS . Then $\varrho/aS = \tau \cap \sigma \supseteq \gamma_a \cap \beta_a$. Since a is arbitrary, $\varrho \supseteq \gamma_a \cap \beta_a$ for all $a \in S$. But π is the smallest congruence containing $\gamma_a \cap \beta_a$ for all $a \in S$, hence $\varrho \supseteq \pi$. Thus π is the minimum completely simple congruence on S .

(2. 8) **Theorem.** *Let S be a simple semigroup with a minimal right ideal, and π be the minimum completely simple congruence on S . Then if ϱ is a regular congruence on S , $\pi \subseteq \varrho$, and ϱ/π is a congruence on S/π . Let θ be the map of $\mathbf{C}$, the lattice of regular congruences on S , to $\mathbf{C}'$, the lattice of congruences on S/π , defined by $\varrho\theta = \varrho/\pi$. Then θ is a lattice isomorphism of $\mathbf{C}$ onto $\mathbf{C}'$.*

The lattice of congruences on a completely simple semigroup, hence on S/π , is discussed in detail in [3]. Two immediate consequences of (2. 8) are:

(2. 9) *Corollary.* The lattice of regular congruences on a simple semigroup with a minimal right ideal is semimodular.

(2. 10) *Corollary.* Let S be a simple semigroup with a minimal right ideal. Then S has a minimum group congruence.

We now generalize (2. 5).

(2. 11) *Corollary.* Let S be a simple semigroup with a minimal right ideal. Then ϱ is a regular congruence on S , for which the idempotents of S/ϱ form a unitary subsemigroup of S/ϱ if and only if $\varrho = \tau \cap \sigma$, where τ is a group congruence on S and σ is a band congruence on S . Moreover, τ and σ are uniquely determined by ϱ .

PROOF. If ϱ is a regular congruence with the idempotents of S/ϱ forming a unitary subsemigroup, then $\pi \subseteq \varrho$. But then there exists ϱ' , a congruence on S/π , such that $\varrho' = \varrho/\pi$. We know that S/π is regular, and hence by (2. 3), it follows that $\varrho' = \tau' \cap \sigma'$, where $\tau'[\sigma']$ is a group [band] congruence on S/π . In fact, τ' and σ' are uniquely determined by ϱ' . But then there exist congruences τ and σ on S containing π that satisfy $\tau' = \tau/\pi$ and $\sigma' = \sigma/\pi$, by ([1] Theorem 1. 6). Clearly $\tau[\sigma]$ is a group [band] congruence on S , and $\varrho = \tau \cap \sigma$. Moreover, since τ' and σ' are uniquely determined by ϱ' , we have τ and σ are uniquely determined by ϱ .

Conversely, suppose that $\varrho = \tau \cap \sigma$, where $\tau[\sigma]$ is a group [band] congruence on S . Then we have $\pi \subseteq \tau$ and $\pi \subseteq \sigma$, so that by ([1] Theorem 1. 6) we have $\tau' = \tau/\pi$ [$\sigma' = \sigma/\pi$] is defined and is a group [band] congruence on S/π . Clearly $(S/\pi)/(\tau' \cap \sigma')$ is isomorphic to S/ϱ . But S/π is regular, and applying (2. 3), we see that the idempotents of $(S/\pi)/(\tau' \cap \sigma')$ form a unitary subsemigroup. Thus the idempotents of S/ϱ form a unitary subsemigroup of S/ϱ , and we have our result.

We close by noting that since the semigroup, S , of example (1. 10) is a simple, idempotent-free semigroup, $\mathbf{R}$ is a band congruence on S ([1] ex. 1, 93, Vol. II). But S has a non-trivial group congruence, ϱ , therefore by (2. 11), $\varrho \cap \mathbf{R}$ is a non-trivial, completely simple congruence on S . For other examples, see [4].

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