

An integrability theorem for power series

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1. HEYWOOD [2] proved the following result:

Theorem A. Suppose that $F(x) = \sum_0^\infty c_n x^n$ for $0 \leq x < 1$, $\gamma < 1$ and that there is a positive number ε such that $c_n > \frac{-K}{n^{\gamma+\varepsilon}}$, for all sufficiently large values of n . (K some positive constant.) Then $(1-x)^{-\gamma} F(x) \in L(0, 1)$ iff $\sum n^{\gamma-1} |c_n|$ converges.

The object of this present note is to obtain a generalization of Theorem A.

2. We prove the following result:

Theorem. Let

$$F(x) = \sum_0^\infty c_n x^n, \quad 0 \leq x < 1 \text{ and } \gamma < 1.$$

Suppose that there is a positive number ε such that

$$(2.1) \quad c_n > \frac{-K}{n^{(\gamma/p) + (1-1/p) + \varepsilon}} \quad (0 < p \leq \infty, K \text{ constant})$$

for all sufficiently large values of n . Then $(1-x)^{-\gamma} (F(x))^p \in L(0, 1)$ iff

$$\sum n^{\gamma-2} \left(\sum_{k=1}^n |c_k| \right)^p$$

converges.

It may be remarked that if we put $p=1$ in our theorem, we get Theorem A.

3. We require the following lemmas.

Lemma 1. [5, p. 58]. If b is a constant, then

$$\frac{\Gamma(x)}{\Gamma(x+b)} \sim x^{-b}, \quad \text{as } x \rightarrow \infty.$$

Lemma 2. [3]. Let

$$F(x) = \sum_0^\infty c_n x^n, \quad c_n \geq 0, \quad 0 \leq x < 1, \quad s_n = \sum_{k=1}^n c_k,$$

and $\gamma < 1$. Then, for $0 < p \leq \infty$,

$$\left(\int_0^1 (1-x)^{-\gamma} (F(x))^p dx \right)^{1/p} < \infty \quad \text{iff} \quad \left(\sum_1^\infty n^{\gamma-2} s_n^p \right)^{1/p} < \infty.$$

Lemma 3. [1, p. 255]. If $c > 1$,

$$s_n = \sum_{k=1}^n a_k, \quad a_k \geq 0,$$

then $\sum n^{-c} s_n^p \leq K \sum n^{-c} (na_n)^p$ ($p \geq 1$).

Lemma 4. [4, p. 83]. If $c > 1$, $0 < p < 1$, $a_n \geq 0$ and $\{n^{-j} a_n\}$ is monotonic decreasing for some $j > 0$, then

$$\sum n^{-c} \left(\sum_1^n a_k \right)^p \leq K \sum n^{-c} (na_n)^p.$$

4. PROOF OF THE THEOREM. We may suppose without loss that $\frac{\gamma}{p} - \frac{1}{p} + \varepsilon$ is not an integer. This will ensure the existence of the following gamma function at all relevant points. Let

$$G(x) \equiv 2K\Gamma\left(\frac{1-\gamma}{p} - \varepsilon\right)(1-x)^{(\gamma-1)/p+\varepsilon} = \sum_0^\infty a_n x^n, \quad \text{for } 0 \leq x < 1.$$

Then, since

$$(1-x)^{(\gamma-1)/p+\varepsilon} = \sum_0^\infty \frac{\Gamma\left(n + \frac{1-\gamma}{p} - \varepsilon\right)}{\Gamma(n+1)\Gamma\left(\frac{1-\gamma}{p} - \varepsilon\right)} x^n,$$

we have

$$(4.1) \quad a_n = 2K \frac{\Gamma\left(n + \frac{1-\gamma}{p} - \varepsilon\right)}{\Gamma(n+1)} \sim \frac{2K}{n^{(\gamma-1)/p+1+\varepsilon}}, \quad \text{as } n \rightarrow \infty,$$

by Lemma 1. It follows from (2.1) that $c_n + a_n$ is positive for all sufficiently large values of n . Since

$$F(x) + G(x) = \sum_0^\infty (c_n + a_n) x^n$$

for $0 \leq x < 1$. Lemma 2 now shows that $(1-x)^{-\gamma} (F(x) + G(x))^p \in L(0, 1)$ iff

$$\sum n^{\gamma-2} \left(\sum_{k=1}^n (a_k + c_k) \right)^p$$

converges.

But $(1-x)^{-\gamma} G^p(x)$ is a multiple of $(1-x)^{ep-1}$ and is therefore integrable L in $(0, 1)$. Moreover (4. 1) shows that $\sum n^{\gamma-2} \left(\sum_1^n a_k \right)^p$ is convergent by Lemmas 3 and 4. Therefore it follows that $(1-x)^{-\gamma} (F(x))^p \in L(0, 1)$ iff

$$\sum n^{\gamma-2} \left(\sum_1^n |c_k| \right)^p < \infty$$

Thus the theorem is proved.

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