

Laplace—Stieltjes transform of Mikusinski operator functions

By LARRY W. WILSON

I.

Introduction

The purpose of this paper is to define and develop an integral which can be used to generalize the Laplace—Stieltjes transform. This generalization will apply to functions of a real variable which take their values in Mikusinski's [4] operator space.

In [1] and [2] GESZTELYI defines Stieltjes integrals of operator functions. In [1] he begins with the integral of f with respect to g where f is operator-valued and g is numerical-valued and in [2] he defines an integral as the limit of a summation. Each of these approaches seems indirect in trying to generalize the Laplace—Stieltjes transform; hence we will define an integral of f with respect to g where f is numerical-valued and g is operator-valued.

1. The integral over a bounded interval

Definition 1. A function $g(\lambda, t)$ is of class H on $[a, b] \times [0, \infty)$ if for each λ in $[a, b]$, $g(\lambda, t)$ is continuous in t on $[0, \infty)$ and for each $T > 0$ there is an M such that if $0 \leq t_0 \leq T$ then the variation of $g(\lambda, t_0)$ on $[a, b]$ is less than M .

Theorem 1. If $f(\lambda)$ is continuous on $[a, b]$ and $g(\lambda, t)$ is of class H on $[a, b] \times [0, \infty)$ then $h(t) = \int_a^b f(\lambda) d_\lambda g(\lambda, t)$ is continuous on $[0, \infty)$.

PROOF. Follows from a theorem in Hildebrandt [3, p. 78].

Definition 2. Operator function g is of class H on $[a, b]$ if there exists an operator P such that $g(\lambda) = P\{q(\lambda, t)\}$ on $[a, b]$ and $q(\lambda, t)$ is of class H on $[a, b] \times [0, \infty)$. We say $P\{q(\lambda, t)\}$ is a representation for $g(\lambda)$ on $[a, b]$.

Definition 3. If $f(\lambda)$ is continuous on $[a, b]$ and $g(\lambda)$ is of class H on $[a, b]$ with representation $P\{q(\lambda, t)\}$ then

$$\int_a^b f(\lambda) dg(\lambda) = P \left\{ \int_a^b f(\lambda) d_\lambda q(\lambda, t) \right\}$$

Theorem 2. If $f(\lambda)$ is continuous and $g(\lambda)$ of class H on $[a, b]$ then $\int_a^b f(\lambda) dg(\lambda)$ exists and is unique.

PROOF. Existence follows from Theorem 1. For uniqueness we let $g(\lambda)$ have representations $P_1\{q_1(\lambda, t)\}$ and $P_2\{q_2(\lambda, t)\}$ on $[a, b]$. Without loss of generality we assume P_1 and P_0 are of class C (i.e. continuous on $[0, \infty)$). For $i=1, 2$ we have $q_i(\lambda, t)$ is continuous in t for $t \geq 0$ for each λ in $[a, b]$ and it is easily verified that for each $t > 0$, $P_i(t - \mathcal{T})q_i(\lambda, \mathcal{T})$ is of uniformly bounded variation in λ over $[a, b]$ for $\mathcal{T}$ in $[0, t]$. Therefore by a theorem in WIDDER [5, p. 25] we have

$$\int_0^t P_i(t - \mathcal{T}) d\mathcal{T} \int_a^b f(\lambda) d_\lambda q_i(\lambda, \mathcal{T}) = \int_a^b f(\lambda) d_\lambda \int_0^t P_i(t - \mathcal{T}) q_i(\lambda, \mathcal{T}) d\mathcal{T}, \quad \text{for } i=1, 2.$$

Therefore,

$$\begin{aligned} P_1 \left\{ \int_a^b f(\lambda) d_\lambda q_1(\lambda, t) \right\} &= \int_0^t P_1(t - \mathcal{T}) \left[\int_a^b f(\lambda) d_\lambda q_1(\lambda, \mathcal{T}) \right] d\mathcal{T} = \\ &= \int_a^b f(\lambda) d_\lambda \int_0^t P_1(t - \mathcal{T}) q_1(\lambda, \mathcal{T}) d\mathcal{T} = \\ &= \int_a^b f(\lambda) d_\lambda \int_0^t P_2(t - \mathcal{T}) q_2(\lambda, \mathcal{T}) d\mathcal{T} = \\ &= P_2 \left\{ \int_a^b f(\lambda) d_\lambda q_2(\lambda, t) \right\}. \end{aligned}$$

Lemma 1. If $q(\lambda, t)$ is of class H on $[a, b] \times [0, \infty)$ and $x(\mathcal{T})$ is continuous on $[0, \infty)$ then $h(\lambda, t) = \int_0^t x(t - \mathcal{T}) q(\lambda, \mathcal{T}) d\mathcal{T}$ is of class H on $[a, b] \times [0, \infty)$.

PROOF. Let P be any partition of $[a, b]$, $t > 0$ and $\mathcal{T} > t$

$$\begin{aligned} \sum_P |h(\lambda_i, t) - h(\lambda_{i-1}, t)| &\leq \sum_P \int_0^t |x(t - \mathcal{T})| |q(\lambda_i, \mathcal{T}) - q(\lambda_{i-1}, \mathcal{T})| d\mathcal{T} \leq \\ &\leq M \sum_P \int_0^t |q(\lambda_i, \mathcal{T}) - q(\lambda_{i-1}, \mathcal{T})| d\mathcal{T} \leq \\ &\leq M \int_0^t \left(\sum_P |q(\lambda_i, \mathcal{T}) - q(\lambda_{i-1}, \mathcal{T})| \right) d\mathcal{T} \leq \\ &\leq MV \int_0^t d\mathcal{T} \leq \\ &\leq MVT \end{aligned}$$

Where $M = \sup |x(\mathcal{T})|$ for $\mathcal{T}$ in $[0, T]$ and V is larger than the variation of $q(\lambda, \mathcal{T})$

in λ over $[a, b]$ for $\mathcal{T}$ in $[0, T]$. Further for each λ in $[a, b]$, $h(\lambda, t)$ is continuous on $[0, \infty)$ since if $\mathcal{T} \equiv t > u > 0$ and $x(u - \mathcal{T}) = x(t - \mathcal{T}) + \varepsilon(\mathcal{T}, u)$ for all $\mathcal{T}$ in $[0, t]$ then

$$\begin{aligned} |h(\lambda, t) - h(\lambda, u)| &= \left| \int_u^t x(t - \mathcal{T}) q(\lambda, \mathcal{T}) d\mathcal{T} - \int_0^u \varepsilon(\mathcal{T}, u) q(\lambda, \mathcal{T}) d\mathcal{T} \right| \leq \\ &\leq M_x M_q \int_u^t d\mathcal{T} + M_q \int_0^u |\varepsilon(\mathcal{T}, u)| d\mathcal{T}, \end{aligned}$$

where $M_x = \sup |x(t)|$ for t in $[0, T]$ and $M_q \equiv |q(\lambda, t)|$ for (λ, t) in $[a, b] \times [0, T]$. Since $\varepsilon(\mathcal{T}, u) \rightarrow 0$ as $u \rightarrow t$ uniformly for u, t in $[0, T]$, we have left continuity for $h(\lambda, t)$ at each t . By similar proof we obtain right continuity. Therefore $h(\lambda, t)$ is of class H on $[a, b] \times [0, \infty)$.

Theorem 3. If $f(\lambda)$, $f_1(\lambda)$ and $f_2(\lambda)$ are continuous on $[a, b]$, $g(\lambda)$, $g_1(\lambda)$, $g_2(\lambda)$ are of class H on $[a, b]$, c an operator, and k a complex number then

$$a) \int_a^b f(\lambda) dcg(\lambda) = c \int_a^b f(\lambda) dg(\lambda)$$

$$b) \int_a^b kf(\lambda) dg(\lambda) = k \int_a^b f(\lambda) dg(\lambda)$$

$$c) \int_a^b f(\lambda) dg(\lambda) = \int_a^r f(\lambda) dg(\lambda) + \int_r^b f(\lambda) dg(\lambda) \quad \text{for } a < r < b.$$

$$d) \int_a^b (f_1(\lambda) + f_2(\lambda)) dg(\lambda) = \int_a^b f_1(\lambda) dg(\lambda) + \int_a^b f_2(\lambda) dg(\lambda)$$

and

$$e) \int_a^b f(\lambda) d(g_1(\lambda) + g_2(\lambda)) = \int_a^b f(\lambda) dg_1(\lambda) + \int_a^b f(\lambda) dg_2(\lambda).$$

PROOF. The proofs of parts b , c , and d are trivial. Part a follows from the fact that if $p\{q(\lambda, t)\}$ is a representation for $g(\lambda)$ then $cp\{q(\lambda, t)\}$ is a representation for $cg(\lambda)$.

To prove part e we let $g_i(\lambda) = p_i\{q_i(\lambda, t)\}$ be a representation for $g(\lambda)$ on $[a, b]$ for $i=1, 2$ where $p_i = a_i/c$ where a_i and c are of class C (continuous on $[0, \infty)$) and $c \neq 0$. By Lemma 1 we have $a_i\{q_i(\lambda, t)\}$ is of class H on $[a, b] \times [0, \infty)$ for $i=1, 2$.

Therefore $g_1(\lambda) + g_2(\lambda)$ has a representation of $\frac{1}{c}\{a_1\{q_1(\lambda, t)\} + a_2\{q_2(\lambda, t)\}\}$ on $[a, b]$ since class H is closed under addition. The conclusion follows.

We note that the integral is a generalization of the Riemann—Stieltjes integral.

That is if $g(\lambda) = \{q(\lambda, t)\}$ where $q(\lambda, t)$ is of class H on $[a, b]$ then $\int_a^b f(\lambda) dg(\lambda) =$

$$= \left\{ \int_a^b f(\lambda) d_x q(\lambda, t) \right\} \text{ for all } f \text{ continuous on } [a, b].$$

2. Limits

We use the following definition of limit which is similar to that of MIKUSINSKI [4] for sequences of operators.

Definition 4. For $F(b)$ an operator function, $\lim_{b \rightarrow \infty} F(b) = a$ if there exists an operator q such that $F(b) = q \{f(b, t)\}$ for all $b > N$ for some $N > 0$, where $f(b, t)$ is continuous in t on $[0, \infty)$ for each $b > N$ and $\lim_{b \rightarrow \infty} f(b, t) = h(t)$ almost uniformly on $[0, \infty)$.

This limit has the same properties as Mikusinski's sequential limits. In particular it is unique and linear.

3. The improper Integral

Definition 5. Operator function $g(\lambda)$ is of class H on $[a, \infty)$ if $g(\lambda) = p \{q(\lambda, t)\}$ where p is an operator and $q(\lambda, t)$ is of class H on $[a, b] \times [0, \infty)$ for all $b > a$. $p \{q(\lambda, t)\}$ is called a representation of $g(\lambda)$ on $[a, \infty)$.

Definition 6. If $f(\lambda)$ is continuous on $[a, \infty)$ and $g(\lambda)$ is of class H on $[a, \infty)$ then

$$\int_a^\infty f(\lambda) dg(\lambda) = \lim_{b \rightarrow \infty} \int_a^b f(\lambda) dg(\lambda)$$

provided the limit exists.

We note that Theorem 3 is easily extended to allow $b = \infty$.

Theorem 4. If $f(\lambda)$ is continuous on $[a, \infty)$, $g(\lambda)$ is of class H on $[a, \infty)$ and $\int_a^\infty f(\lambda) dg(\lambda)$ exists then $g(\lambda)$ has a representation $p \{q(\lambda, t)\}$ on $[a, \infty)$ such that $\lim_{b \rightarrow \infty} \int_a^b f(\lambda) dq(\lambda, t)$ exists almost uniformly and $\int_a^b f(\lambda) dg(\lambda) = p \left\{ \int_a^\infty f(\lambda) d_\lambda q(\lambda, t) \right\}$.

PROOF. Let $c \{q(\lambda, t)\}$ be a representation of $g(\lambda)$ on $[a, \infty)$. Then $\int_a^\infty f(\lambda) dg(\lambda) = \lim_{b \rightarrow \infty} c \left\{ \int_a^b f(\lambda) d_\lambda q(\lambda, t) \right\}$ implies the existence of an operator p and a function $F(b, t)$ continuous in t on $[0, \infty)$ for each $b > N$ for some $N > 0$ such that $\lim_{b \rightarrow \infty} F(b, t) = h(t)$ almost uniformly on $[0, \infty)$ and $c \left\{ \int_a^b f(\lambda) d_\lambda q(\lambda, t) \right\} = p \{F(b, t)\}$. Let $c = c_1/e$ and $p = p_1/e$ where c_1 , p_1 and e are of class C (continuous on $[0, \infty)$) and $e \neq 0$. Then $c_1 \left\{ \int_a^b f(\lambda) d_\lambda q(\lambda, t) \right\} = p_1 \{F(b, t)\}$. Clearly $\lim_{b \rightarrow \infty} p_1 \{F(b, t)\} = p_1 \{h(t)\}$ almost uniformly on $[0, \infty)$. Therefore

$$\begin{aligned}
\int_a^\infty f(\lambda) dg(\lambda) &= p_1/e \{h(t)\} = \\
&= p_1/e \left\{ \lim_{b \rightarrow \infty} F(b, t) \right\} = \\
&= 1/e \left\{ \lim_{b \rightarrow \infty} p_1 \{F(b, t)\} \right\} = \\
&= 1/e \left\{ \lim_{b \rightarrow \infty} c_1 \left\{ \int_a^b f(\lambda) d_\lambda q(\lambda, t) \right\} \right\} = \\
&= 1/e \left\{ \lim_{b \rightarrow \infty} \int_a^b f(\lambda) d_\lambda c_1 \{q(\lambda, t)\} \right\}.
\end{aligned}$$

Since $q(\lambda, t)$ is of class H on $[a, \infty) \times [0, \infty)$, $c_1 \{q(\lambda, t)\}$ is of class H on $[a, \infty) \times [0, \infty)$ by Lemma 1. Therefore $1/e \{c_1 \{q(\lambda, t)\}\}$ is the desired representation of $g(\lambda)$ on $[a, \infty)$.

4. The transform

Definition 7. If $f(\lambda) = f_1(\lambda) + if_2(\lambda)$ and $g(\lambda)$ is an operator function we take

$$\int_a^b f(\lambda) dg(\lambda) = \int_a^b f_1(\lambda) dg(\lambda) + i \int_a^b f_2(\lambda) dg(\lambda)$$

provided $f_1(\lambda)$ and $f_2(\lambda)$ are integrable with respect to $g(\lambda)$. We allow $b = \infty$ in this definition.

Definition 8. If $g(\lambda)$ is of class H on $[0, \infty)$ and r is a complex number such that $\int_0^\infty e^{-\lambda r} dg(\lambda)$ exists then we call this integral the Laplace—Stieltjes transform of $g(\lambda)$ and denote it as $L(g(\lambda))$.

That the transform is well defined linear, and a generalization of the usual Laplace—Stieltjes transform follows from the properties of the improper integral.

Using the definition of MIKUSINSKI for the continuous derivative $g'(\lambda)$ of operator function $g(\lambda)$ for interval $[0, \infty)$ we have the following theorem.

Theorem 5. If $g(\lambda)$ has a representation $p \{q(\lambda, t)\}$ on $[0, \infty)$ such that $q_\lambda(\lambda, t)$ is continuous on $[0, \infty) \times [0, \infty)$, $\int_0^\infty e^{-r\lambda} d_\lambda g_\lambda(\lambda, t)$ converges almost uniformly for $0 \leq t < \infty$ and $\lim_{\lambda \rightarrow \infty} e^{-\lambda r} q_\lambda(\lambda, t) = 0$ almost uniformly for t in $[0, \infty)$ then

$$L(g'(\lambda)) = -g'(0) + rL(g(\lambda)).$$

PROOF. $\int_0^\infty e^{-\lambda r} d_\lambda q_\lambda(\lambda, t) =$

$$= \lim_{b \rightarrow \infty} \left[e^{-\lambda r} q_\lambda(\lambda, t) \Big|_0^b + r \int_0^b e^{-\lambda r} q_\lambda(\lambda, t) d_\lambda \right]$$

$$= -q_\lambda(0, t) + r \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)$$

$\int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)$ exist almost uniformly on $[0, \infty)$ since $\int_0^\infty e^{-\lambda r} d_\lambda q_\lambda(\lambda, t)$ and $\lim_{b \rightarrow \infty} e^{-\lambda b} q_\lambda(b, t)$ exist almost uniformly. Hence

$$L(g'(\lambda)) = p \left\{ -q_\lambda(0, t) + r \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t) \right\} = -g'(0) + rL(g(\lambda)).$$

5. Transforms of specific function

Consider the heaviside function

$$H_\lambda(t) = \begin{cases} 0, & 0 \leq t \leq \lambda \\ 1, & \lambda < t \end{cases}.$$

Theorem 6. If $H(\lambda) = \{H_\lambda(t)\}$ then $L(H(\lambda)) = \frac{-1}{s+r}$.

PROOF. $H_\lambda = s\{h_1(\lambda, t)\}$ where $h_1(\lambda, t) = \begin{cases} 0, & 0 \leq t \leq \lambda \\ t - \lambda, & \lambda < t \end{cases}$.

Clearly $h_1(\lambda, t)$ is of class H on $[0, b] \times [0, \infty)$ for all $b > 0$, therefore

$$\int_0^b e^{-\lambda r} d_\lambda H(\lambda) = s \left\{ \int_0^b e^{-\lambda r} d_\lambda h_1(\lambda, t) \right\} = s \{g_b(r, t)\}$$

where

$$g_b(r, t) = \begin{cases} \frac{e^{-tr} - 1}{r}, & t \leq b \\ \frac{e^{-br} - 1}{r}, & b < t \end{cases}$$

$g_b(r, t)$ is continuous for t in $[0, \infty)$ if $r \neq 0$ $b > 0$. Further for $T > 0$ if $b > T$ and t in $[0, T]$ then

$$g_b(r, t) = \frac{e^{-tr} - 1}{r}$$

and $g_b(r, t)$ has a uniform limit as $b \rightarrow \infty$. Therefore

$$\int_0^\infty e^{-\lambda r} d_\lambda H(\lambda) = s \left\{ \frac{e^{-tr} - 1}{r} \right\} = -\{e^{-tr}\} = \frac{-1}{s+r}$$

if $r \neq 0$. If $r = 0$ for $b > 0$ we have

$$\int_0^b e^{-\lambda r} dH(\lambda) = \int_0^b dH(\lambda) = s\{f_b(t)\}$$

where $f_b(t) = \begin{cases} -t, & t \leq b \\ -b, & b < t \end{cases}$ is continuous for $b > 0$, and $t > 0$ and $\lim_{b \rightarrow \infty} f_b(t) = -t$ almost uniformly on $[0, \infty)$. Therefore

$$\int_0^b dH(\lambda) = s\{-t\} = -\{1\} = \frac{-1}{s} = \frac{-1}{s+r}$$

since $r=0$.

Next we consider the translation operator $h(\lambda) = s\{H_\lambda(t)\}$.

Corollary 8a. $L(h(\lambda)) = \frac{-s}{s+r}$ for all r .

PROOF. Follows from the linearity of the transform and Theorem 6.

6. Application

We define the inverse transform by $L^{-1}(h(r)) = g(\lambda)$ if $L(g(\lambda)) = h(r)$, and note that the transform may be used to solve operational differential equations.

As an example consider $x''(\lambda) = -sx'(\lambda)$. Proceeding formally and using Theorem 5 we obtain $L(x(\lambda)) = \frac{A}{r} + \frac{B}{r+s}$ where A and B depend on $x'(0)$ and $x''(0)$. Taking the inverse transform we obtain $x(\lambda) = A\lambda - BH(\lambda)$.

II.

Introduction

In I. we defined a Laplace—Stieltjes transform for operator valued functions of a complex variable. We will extend this work further by considering convergence and operational properties for the transform.

1. Convergence theory and order properties

Notation

If we write $L(g(\lambda)) = p\left\{\int_0^\infty e^{-\lambda t} dq(\lambda, t)\right\}$ exists from some r then $g(\lambda) = p\{q(\lambda, t)\}$ for $\lambda \in [0, \infty)$, where p is an operator, $q(\lambda, t) \in H$ on $[0, \infty) \times [0, \infty)$ (i.e. on $[0, b] \times [0, \infty)$ for all $b > 0$), and $\int_0^\infty e^{-\lambda t} dq(\lambda, t)$ exists almost uniformly with respect to t on $[0, \infty)$.

We note that the existence of the transform implies the existence of operator p and function $q(\lambda, t)$ with the above properties by Theorem 4 and Definition 8 in I.

We have the following generalization of the usual theorem concerning the region of convergence of the transform.

Theorem 1. If $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} dq(\lambda, t) \right\}$ exists for $r_0 = a_0 + ib_0$ then $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} dq(\lambda, t) \right\}$ exists for all $r = a + ib$ with $a > a_0$.

PROOF. Let $B(u, t) = \int_0^u e^{-\lambda r_0} dq(\lambda, t)$ and $T > 0$. Since $\lim_{u \rightarrow \infty} B(u, t) = h(t)$ uniformly on $[0, T]$ and $q(\lambda, t)$ is of uniformly bounded variation in λ on any interval of the form $[0, N]$ for t in $[0, T]$ and continuous in t on $[0, T]$ for each $\lambda \geq 0$, there exists a $J > 0$ such that $|B(b, t)| < J$ for all $(b, t) \in [0, \infty) \times [0, T]$. Further,

$$\begin{aligned} \int_0^b e^{-\lambda r} d_\lambda q(\lambda, t) &= \int_0^b e^{-\lambda(r-r_0)} d_\lambda B(\lambda, t) = \\ &= e^{-b(r-r_0)} B(b, t) + (r-r_0) \int_0^b e^{-\lambda(r-r_0)} B(\lambda, t) d\lambda. \end{aligned}$$

Since $a > a_0$ and $|B(b, t)| < J$ for $(b, t) \in [0, \infty) \times [0, T]$, we have $e^{-b(r-r_0)} B(b, t) \rightarrow 0$ uniformly on $[0, T]$ as $b \rightarrow \infty$.

Also for $t \in [0, T]$,

$$\left| \int_b^\infty e^{-\lambda(r-r_0)} B(\lambda, t) d\lambda \right| \leq J \int_b^\infty e^{-\lambda(r-r_0)} d\lambda \leq \frac{J}{a-a_0} e^{-b(a-a_0)},$$

which approaches 0 as $b \rightarrow \infty$. Therefore $\int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)$ converges almost uniformly on $[0, \infty)$ for $a > a_0$.

Theorem 2. If $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} dq(\lambda, t) \right\}$ exists for $r = a + ib$ then

(1) $a > 0$ implies $\lim_{\lambda \rightarrow \infty} q(\lambda, t) e^{-a\lambda} = 0$ almost uniformly on $[0, \infty)$

and

(2) $a < 0$ implies $\lim_{\lambda \rightarrow \infty} q(\lambda, t) = q(\infty, t)$ exists and $\lim_{\lambda \rightarrow \infty} (q(\infty, t) - q(\lambda, t)) e^{-a\lambda} = 0$ almost uniformly on $[0, \infty)$.

PROOF. If $a > 0$ let $B(u, t) = \int_0^u e^{-rw} d_w q(w, t)$ then

$$q(\lambda, t) - q(0, t) = \int_0^\lambda d_\lambda q(\lambda, t) = \int_0^\lambda e^{ru} d_u B(u, t).$$

Integrating by parts, multiplying by $e^{-r\lambda}$ rearranging and taking the limit of both sides of the equation gives us

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} (q(\lambda, t) - q(0, t))e^{-r\lambda} &= B(\infty, t) - \lim_{\lambda \rightarrow \infty} re^{-\lambda r} \int_0^\lambda e^{ru} B(u, t) du = \\ &= \lim_{\lambda \rightarrow \infty} re^{-r\lambda} \int_0^\lambda e^{ru} [B(\infty, t) - B(u, t)] du.\end{aligned}$$

Since $\lim_{\lambda \rightarrow \infty} B(\lambda, t) = B(\infty, t)$ exists almost uniformly on $[0, \infty)$, for $\varepsilon > 0$, $T > 0$ there exists a J such that if $\lambda \geq J$ then

$$|B(\lambda, t) - B(\infty, t)| < \varepsilon \quad \text{for all } t \in [0, T]. \quad \text{For } \lambda > J,$$

$$\int_0^\lambda e^{ru} (B(\infty, t) - B(u, t)) du = \int_0^J e^{ru} (B(\infty, t) - B(u, t)) du + \int_J^\lambda e^{ru} (B(\infty, t) - B(u, t)) du.$$

However,

$$\left| re^{-r\lambda} \int_0^J e^{ru} (B(\infty, t) - B(u, t)) du \right| \leq |r| e^{-\lambda a} e^{aJ} MJ,$$

where M is an upper bound of $|B(\infty, t) - B(u, t)|$ for $0 \leq u \leq J$, $0 \leq t \leq T$. Also,

$$\left| \int_J^\lambda e^{ru} (B(\infty, t) - B(u, t)) du \right| \leq \int_J^\lambda e^{au} \varepsilon du \leq \frac{e^{a\lambda}}{a} \varepsilon.$$

Choose $K > J$ such that if $\lambda > K$ then $|r| e^{-a\lambda} e^{aJ} MJ < \varepsilon$. Then for $\lambda > K$, $t \in [0, T]$ we have

$$\left| re^{-\lambda r} \int_0^\infty e^{ru} (B(\infty, t) - B(u, t)) du \right| \leq \varepsilon + \frac{|r|}{a} \varepsilon.$$

Therefore $\lim_{\lambda \rightarrow \infty} (q(\lambda, t) - q(0, t))e^{-\lambda r} = 0$ and hence $\lim_{\lambda \rightarrow \infty} q(\lambda, t)e^{-\lambda r} = 0$.

If $a < 0$ by Theorem 1 we have $L(g(\lambda))$ exists at $r=0$ or $\int_0^\infty d_\lambda q(\lambda, t) = q(\infty, t) + (-q(0, t))$ almost uniformly for $t \in [0, \infty)$. Therefore $q(\infty, t)$ exists. Further since $a < 0$.

$$q(\infty, t) - q(\lambda, t) = \int_\lambda^\infty d_u q(u, t) = \int_\lambda^\infty e^{ru} d_u B(u, t) = e^{r\lambda} B(\lambda, t) - r \int_\lambda^\infty e^{ru} B(u, t) du.$$

Therefore

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} (q(\infty, t) - q(\lambda, t))e^{-r\lambda} &= B(\infty, t) - \lim_{\lambda \rightarrow \infty} re^{-\lambda r} \int_\lambda^\infty e^{ru} B(u, t) du = \\ &= \lim_{\lambda \rightarrow \infty} re^{-\lambda r} \int_\lambda^\infty e^{ru} (B(\infty, t) - B(u, t)) du.\end{aligned}$$

For $\varepsilon > 0$, $T > 0$ we choose J as before such that if $\lambda \geq J$,

$|B(\infty, t) - B(\lambda, t)| < \varepsilon$ for $0 \leq t \leq T$. Then for $\lambda \geq J$,

$$\left| re^{-r\lambda} \int_{\lambda}^{\infty} e^{ru} (B(\infty, t) - B(u, t)) du \right| \leq |r| e^{-a\lambda} \varepsilon \int_{\lambda}^{\infty} e^{au} du \leq \frac{|r|}{|a|} \varepsilon.$$

Therefore $\lim_{\lambda \rightarrow \infty} |(q(\infty, t) - q(0, t))e^{-\lambda r}| = 0$ and hence $\lim_{\lambda \rightarrow \infty} (q(\infty, t) - q(0, t))e^{-\lambda a} = 0$ almost uniformly on $[0, \infty)$.

Definition 1. If a is any real number, $g(\lambda)$ an operator valued function is said to be of order $a^{a\lambda}$ if there is an operator p such that $g(\lambda) = p\{q(\lambda, t)\}$ where $q(\lambda, t) \in H$ on $[0, \infty) \times [0, \infty)$ and $|q(\lambda, t)| \leq m(t)e^{a\lambda}$ where $m(t)$ is bounded on $[0, T]$ for all $T > 0$.

Theorem 3. If $g(\lambda)$ is an operator function of order $e^{a\lambda}$ then $L(g(\lambda))$ exists for all $r = c + di$, with $c > a$.

PROOF. Let $g(\lambda) = p\{q(\lambda, t)\}$ where $q(\lambda, t)$ has the properties of $q(\lambda, t)$ in definition 1. For $T > 0$,

$$\int_0^R e^{-\lambda r} d_{\lambda} q(\lambda, t) = q(R, t)e^{-Rr} - q(0, t) + r \int_0^R e^{-\lambda r} q(\lambda, t) d\lambda.$$

But $|q(R, t)e^{-rR}| \leq m(t)e^{aR}|e^{-rR}| \leq M_T e^{-(c-a)R}$ where M_T is an upper bound for $m(t)$ on $[0, T]$. Therefore $\lim_{R \rightarrow \infty} q(R, t)e^{-rR} = 0$ uniformly on $[0, T]$. Further

$$\int_0^R |e^{-\lambda r} q(\lambda, t)| d\lambda \leq \int_0^R e^{-\lambda(c-a)} m(t) d\lambda \leq M_T \frac{e^{-R(c-a)} - 1}{-(c-a)}.$$

Therefore $\int_0^{\infty} e^{-\lambda r} q(\lambda, t) d\lambda$ and hence $\int_0^{\infty} e^{-\lambda r} d_{\lambda} q(\lambda, t)$ converge almost uniformly for $t \in [0, \infty)$.

Theorem 4. If $\lim_{\lambda \rightarrow \infty} g(\lambda) = g(\infty)$ exists and $g(\lambda) - g(\infty)$ is of order $e^{a\lambda}$ then $L(g(\lambda))$ exists for all $r = c + id$ with $c > a$.

PROOF. $L(g(\lambda) - g(\infty))$ exists by Theorem 3.

$$L(g(\lambda) - g(\infty)) = \int_0^{\infty} e^{-\lambda r} d(g(\lambda) - g(\infty)) = \int_0^{\infty} e^{-\lambda r} dg(\lambda) = L(g(\lambda)).$$

Theorem 5. If $L(g(\lambda)) = p\left\{\int_0^{\infty} e^{-\lambda r} d_{\lambda} q(\lambda, t)\right\}$ exists for $r = a + bi$ then

- 1) $a > 0$ implies $g(\lambda)$ is of order $e^{a\lambda}$ and
- 2) $a < 0$ implies $g(\lambda) - g(\infty)$ is order $e^{a\lambda}$.

PROOF. By Theorem 2 for $T > 0$ there is a J such that if $\lambda > J$ then for $t \in [0, T]$, $|q(\lambda, t)e^{-a\lambda}| < 1$. Further $|q(\lambda, t)| < M$ for $\lambda \in [0, J]$, $t \in [0, T]$ since $q(\lambda, t)$ is of uniformly bounded variation in λ on $[0, J]$ for $t \in [0, T]$. Let $m(T) = \text{lub}\{M | |q(\lambda, t)| \leq$

$\leq Me^{a\lambda}$ for $\lambda \geq 0, t \in [0, T]$ then $|q(\lambda, t)| \leq m(t)e^{a\lambda}$ and $m(t)$ is bounded on every bounded interval $[0, T]$.

The proof of part 2 is similar and will be omitted.

Theorem 6. If $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} dq(\lambda, t) \right\}$ exists for some $r = a + bi$ and $h(r, t) = \int_0^\infty e^{-\lambda r} dq(\lambda, t)$ then

$$1) \text{ If } a > 0 \quad h(r, t) = r \int_0^\infty e^{-\lambda r} q(\lambda, t) d\lambda - q(0, t) \text{ and } \int_0^\infty e^{-\lambda r} q(\lambda, t) d\lambda$$

converges absolutely and almost uniformly for $t \in [0, \infty)$ for all $r = c + di$ with $c > a$, and

$$2) \text{ if } a < 0 \quad h(r, t) = q(\infty, t) - q(0, t) + r \int_0^\infty e^{-\lambda r} (q(\lambda, t) - q(\infty, t)) d\lambda$$

and $\int_0^\infty e^{-\lambda r} (q(\lambda, t) - q(\infty, t)) d\lambda$ converges absolutely and almost uniformly for $t \in [0, \infty)$ for all $r = c + di$ with $c > a$.

PROOF. If $a > 0$, by Theorem 5, $g(\lambda)$ is of order $e^{a\lambda}$, and in the proof of Theorem 3 we have already proven that $h(r, t) = r \int_0^\infty e^{-\lambda r} q(\lambda, t) d\lambda - q(0, t)$ and that the integral converges absolutely and almost uniformly.

If $a < 0$, after using Theorem 5, the proof is similar to that of Theorem 3 and will be omitted.

Definition 2. If $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t) \right\}$ exists for each $r \in A$ we say $L(g(\lambda))$

exists uniformly on A if $\int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)$ converges uniformly for $r \in A$ and almost uniformly for $t \in [0, \infty)$.

Theorem 7. If $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r_0} d_\lambda q(\lambda, t) \right\}$ exists for $r_0 = a_0 + b_0 i$, $H > 0$ and $K > 1$ then $L(g(\lambda))$ exists uniformly on $A = \{r | r = a + bi \text{ such that } |r - r_0| \leq K(a - a_0)e^{H(a - a_0)} \text{ and } a > a_0\}$.

PROOF. If $r = a + bi \in A$ and $a > a_0$ the $L(g(\lambda))$ exists by Theorem 1 and if $a = a_0$ then $r = r_0$ and $L(g(\lambda))$ exists. For $\varepsilon > 0, T > 0$ we must show the existence of R_0 such that if $R > R_0$

$$\left| \int_R^\infty e^{-\lambda r} d_\lambda q(\lambda, t) \right| < \varepsilon \quad \text{for all } r \in A \text{ and } t \in [0, T].$$

Let $B(\lambda, t) = \int_0^\infty e^{-\lambda r_0} d_\lambda q(\lambda, t)$, since $B(\infty, t)$ exists uniformly on $[0, T]$ pick $R_0 > H$ such that

$$|B(w, t) - B(u, t)| < \varepsilon/K \quad \text{if } w, u > R_0 \text{ for all } t \text{ in } [0, T].$$

For $R > R_0$, $r = a + bi \in A$ with $a > a_0$ we have

$$\begin{aligned} \left| \int_R^\infty e^{-\lambda r} d_\lambda q(\lambda, t) \right| &= \left| \int_R^\infty e^{-(r-r_0)} d_\lambda (B(\lambda, t) - B(R, t)) \right| \leq \\ &\leq |r - r_0| \int_R^\infty e^{-\lambda(r-r_0)} |B(\lambda, t) - B(R, t)| d\lambda \leq \\ &\leq |r - r_0| \varepsilon / K \int_R^\infty e^{-(a-a_0)\lambda} d\lambda \leq \\ &\leq K(a-a_0) e^{H(a-a_0)} \varepsilon / K \frac{e^{-(a-a_0)R}}{a-a_0} \leq \\ &\leq \varepsilon, \end{aligned}$$

since $H < R$ and $a > a_0$. If $a = a_0$ then

$$\left| \int_R^\infty e^{-r\lambda} d_\lambda q(\lambda, t) \right| = |B(\infty, t) - B(R, t)| \leq \varepsilon / K < \varepsilon.$$

The above inequalities are clearly independent of $t \in [0, T]$.

Definition 3. $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t) \right\}$ exists absolutely for some r if $\int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)$ converges absolutely and almost uniformly on $[0, \infty)$.

Theorem 8. If $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t) \right\}$ exists absolutely for $r = a + bi$ then it exists absolutely and uniformly for all $r = c + di$ with $c \geq a$.

PROOF. Let $Vq(\lambda, t)$ be the variation of $q(u, t)$ for $0 \leq u \leq \lambda$ for each $t \geq 0$, then for $T > 0$, $\varepsilon > 0$ there is an R such that $\int_R^\infty e^{-\lambda a} d_\lambda Vq(\lambda, t) < \varepsilon$ for $t \in [0, T]$. Therefore if $c \geq a$, $r = c + di$ then $\int_R^\infty |e^{-\lambda r}| d_\lambda Vq(\lambda, t) < \varepsilon$ for all $t \in [0, T]$.

2. Operational properties

Theorem 9. If $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t) \right\} = h(r)$ exists for some r , $c > 0$ then $L(g(\lambda/c)) = h(cr)$.

PROOF. $g(\lambda/c) = p \{q(\lambda/c, t)\}$ and $q(\lambda/c, t) \in H$ on $[0, \infty) \times [0, \infty)$. Using $uc = \lambda$ we obtain

$$p \left\{ \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda/c, t) \right\} = p \left\{ \int_0^\infty e^{-urc} d_u q(u, t) \right\} = h(cr).$$

Theorem 10. If $L(g(\lambda))=h(r)$ exists for $r=a+bi$ then for $r=c+di$, $c>a$, and k any positive integer

$$h^{(k)}(r) = \int_0^\infty (-\lambda)^k e^{-\lambda r} dg(\lambda).$$

PROOF. Let $L(g(\lambda)) = p \left\{ \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t) \right\}$ exist for $r = a + bi$ and $h_1(r, t) = \int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)$. By Theorem 5 in WIDDER [5] we have $\frac{\partial^k}{\partial r^k} (h_1(r, t)) = \int_0^\infty (-\lambda)^k e^{-\lambda r} d_\lambda (q(\lambda, t))$ for each $t \geq 0$.

We know $\frac{\partial^k h_1}{\partial r^k}$ is continuous as a function of r since the $\frac{\partial^{k+1} h_1}{\partial r^{k+1}}$ exists. We need only show the almost uniform convergence of this integral to get that it is continuous as a function of t .

We consider the convergence problem for $k=1$.

i) If $c > a > 0$ we have

$$\int_0^\infty (-\lambda) e^{-\lambda r} d_\lambda q(\lambda, t) = -\lambda e^{-\lambda r} q(\lambda, t) \Big|_0^\infty - \int_0^\infty q(\lambda, t) d_\lambda (-\lambda e^{-\lambda r}).$$

But $\lambda e^{-\lambda r} q(\lambda, t) \rightarrow 0$ almost uniformly for $t \in [0, \infty)$ as $\lambda \rightarrow \infty$ and

$$-\int_0^\infty q(\lambda, t) d_\lambda (-\lambda e^{-\lambda r}) = \int_0^\infty q(\lambda, t) e^{-\lambda r} d\lambda - r \int_0^\infty \lambda q(\lambda, t) e^{-\lambda r} d\lambda.$$

However $\int_0^\infty q(\lambda, t) e^{-\lambda r} d\lambda$ exists almost uniformly on $[0, \infty)$ by Theorem 6. By

Theorem 5 $\int_R^\infty |\lambda e^{-\lambda r} q(\lambda, t)| d\lambda \leq \int_R^\infty \lambda e^{-\lambda c} m(t) e^{\lambda a} d\lambda$ where $m(t)$ is bounded on $[0, T]$ for any $T > 0$. The integral on the right converges to 0 as $R \rightarrow \infty$ uniformly on $[0, T]$.

Therefore $\int_0^\infty \lambda e^{-\lambda r} q(\lambda, t) d\lambda$ converges almost uniformly on $[0, \infty)$.

ii) If $c > a \geq 0$ then $L(g(\lambda))$ exists for $r=a_1$ for some $0 < a_1 < c$ and the problem reduces to case i).

iii) If $a < 0$ and $a < c$ then

$$\int_0^\infty (-\lambda) e^{-\lambda r} d_\lambda q(\lambda, t) = \int_0^\infty (-\lambda) e^{-\lambda r} d_\lambda (q(\lambda, t) - q(\infty, t))$$

and the proof is as in case i) with $q(\lambda, t)$ replaced by $q(\lambda, t) - q(\infty, t)$.

Using induction and similar arguments to those in the case of $k=1$ it is easily shown that convergence is almost uniform on $[0, \infty)$ for all positive integers k .

Multiplying $\frac{\partial^k h_1}{\partial r^k}$ by p completes the proof.

Definition 4. If $g(\lambda) \in H$ on $[a, b]$, $f(\lambda) \in C[a, b]$ then $\int_a^b g(\lambda) df(\lambda) = g(b)f(b) - g(a)f(a) - \int_a^b f(\lambda) dg(\lambda)$. If $g(\infty)$, $f(\infty)$ and $\int_a^\infty f(\lambda) dg(\lambda)$ exists we allow $b = \infty$ in the above definition (which of course changes $[a, b]$ to $[a, \infty)$).

Theorem 11. If $L(g(\lambda)) = h(r)$ exists for $r_0 = a + bi$ then

$$1) \quad a < 0 \text{ implies } L\left(\int_0^\lambda q(u) du\right) = \frac{h(r_0) + g(0)}{r_0} \text{ for } r = r_0$$

and

$$2) \quad a < 0 \text{ implies } L\left(\int_0^\lambda (g(\lambda) - g(\infty)) d\lambda\right) = \frac{h(r_0) + g(0) - g(\infty)}{r_0} \text{ for } r = r_0.$$

PROOF. Let $h(r_0) = p\{h_1(r_0, t)\}$ where $g(\lambda) = p\{q(\lambda, t)\}$ and

$$\begin{aligned} h_1(r_0, t) &= \int_0^\infty e^{-\lambda r_0} d_\lambda q(\lambda, t) = \\ &= r_0 \int_0^\infty e^{-\lambda r_0} q(\lambda, t) d\lambda - q(0, t) = \\ &= r_0 \int_0^\infty e^{-\lambda r_0} d_\lambda \left[\int_0^\lambda q(\lambda, t) d\lambda \right] - q(0, t), \end{aligned}$$

and

$$\frac{h_1(r_0, t) + q(0, t)}{r_0} = \int_0^\infty e^{-\lambda r_0} d_\lambda \left(\int_0^\lambda q(\lambda, t) d\lambda \right)$$

almost uniformly on $[0, \infty)$. Then

$$\begin{aligned} p \frac{\{h_1(r_0, t) + q(0, t)\}}{r_0} &= \int_0^\infty e^{-\lambda r_0} d\lambda \left(p \left\{ \int_0^\lambda q(\lambda, t) d\lambda \right\} \right) \\ \frac{h(r_0) + g(0)}{r_0} &= \int_0^\infty e^{-\lambda r_0} d_\lambda \left(p \left\{ \lambda q(\lambda, t) - \int_0^\lambda \lambda d_\lambda q(\lambda, t) \right\} \right) = \\ &= \int_0^\infty e^{-\lambda r_0} d_\lambda \left(\lambda g(\lambda) - \int_0^\lambda \lambda dg(\lambda) \right) = \int_0^\infty e^{-\lambda r_0} d \left(\int_0^\lambda g(\lambda) d\lambda \right). \end{aligned}$$

The proof of 2) is similar and will be omitted.

Theorem 12. If $L(g(\lambda)) = h(r)$ for some $r_0 = a + ib$, then

$$1) \quad a \geq 0 \text{ implies } L(\lambda(g(\lambda) - g(\infty))) = \frac{h(r) + g(0)}{r} - h'(r) \text{ for } r = c + di \text{ with}$$

$c > a$ and

2) $a < 0$ implies $L(\lambda(g(\lambda) - g(\infty))) = \frac{h(r) + g(0) - g(\infty)}{r} - h'(r)$ for $r = c + di$ with $a < c < 0$.

PROOF. If $c > a > 0$ let $h(r) = p\left\{\int_0^\infty e^{-\lambda r} dq(\lambda, t)\right\}$, then $\lambda g(\lambda) = p\{\lambda q(\lambda, t)\}$ and

$$\begin{aligned} p\left\{\int_0^\infty e^{-\lambda r} d_\lambda(\lambda q(\lambda, t))\right\} &= p\left\{\int_0^\infty e^{-\lambda r} q(\lambda, t) d\lambda + \int_0^\infty e^{-\lambda r} \lambda d_\lambda q(\lambda, t)\right\} = \\ &= \frac{h(r) + g(0)}{r} - h'(r) \end{aligned}$$

by Theorems 10 and 11.

If $a < 0$ the proof is similar to the above.

Theorem 13. If $L(g(\lambda)) = h(r)$ for $r = a_0 + ib_0$ then

1) $a > 0$ implies $L(e^{a\lambda} g(\lambda)) = \frac{rh(r-a) + ag(0)}{r-a}$ for $\operatorname{Re}(r-a) > a_0$

and

2) $a < 0$ implies $L(e^{a\lambda}(g(\lambda) - g(\infty))) = \frac{rh(r-a) + a[g(0) - g(\infty)]}{r-a}$ for $a_0 < \operatorname{Re}(r-a) < 0$.

PROOF. Let $h(r) = p\left\{\int_0^\infty e^{-\lambda r} dq(\lambda, t)\right\}$ and $a_0 > 0$ then

$$\int_0^\infty e^{-\lambda r} d_\lambda(e^{a\lambda} q(\lambda, t)) = \int_0^\infty e^{-\lambda(r-a)} d_\lambda q(\lambda, t) + a \int_0^\infty e^{-(r-a)\lambda} d_\lambda \left(\int_0^\lambda q(\lambda, t) d\lambda\right).$$

Multiplying by p and using Theorem 11 completes the proof of part 1. The proof of 2) is similar and will be omitted.

References

- [1] V. E. GESZTELYI, Über das Stieltjes-Integral von Operatorfunktionen I: *Publ. Math. Debrecen* **12** (1965), 307—322.
- [2] V. E. GESZTELYI, Über das Stieltjes-Integral von Operatorfunktionen II. *Publ. Math. Debrecen* **14** (1967), 313—324.
- [3] T. H. HILDEBRANDT, *Theory of Integration*. New York 1963.
- [4] J. MIKUSINSKI, *Operational Calculus*. Oxford 1959.
- [5] D. V. WIDDER, *The Laplace Transform*. Princeton.
- [6] L. W. WILSON, Laplace—Stieltjes Transform of Mikusinski Operator Functions, *Ph. D. Dissertation*, University of Texas, 1971.

(Received December 29, 1971.)