

## On the absolute Riesz summability of Fourier series

By ABRAR AHMAD KHAN (Aligarh)

The object of this paper is to generalize a recent Theorem of PREM CHANDRA on absolute Riesz summability of Fourier series. It has also been shown that one of his conditions is redundant.

1. Let  $\sum_{n=0}^{\infty} a_n$  be a given infinite series with the sequence of partial sums  $\{S_n\}$  and let  $\{\lambda_n\}$  be an increasing sequence of positive numbers tending to infinity with  $n$  and

$$\lambda_n = \mu_0 + \mu_1 + \mu_2 + \dots + \mu_n,$$

$$t_n = \frac{1}{\lambda_n} \sum_{v=0}^n \mu_v S_v.$$

A series  $\sum_n a_n$  is said to be *summable*  $[R, \lambda_n, 1]$  if\*)  $t_n \in BV$  and write  $\sum_{n=0}^{\infty} a_n \in [R, \lambda_n, 1]$ .

Let  $f(t)$  be  $L$ -integrable in  $(-\pi, \pi)$  with period  $2\pi$ . Without any loss of generality the constant term of the Fourier series of  $f(t)$  can be taken to be zero, so that

$$\int_{-\pi}^{\pi} f(t) dt = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We shall use the following notations:

$$\Phi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\},$$

$$A(t) = \frac{1}{t} \int_0^t u d\Phi(u),$$

$$K(n, t) = \sum_{v=0}^n \frac{\lambda_v}{v+1} \sin vt.$$

\*)  $t_n \in BV$  means  $\sum |t_n - t_{n-1}| < \infty$ .

2. Recently PREM CHANDRA [1] has proved the following theorem.

**Theorem A.** Let for  $0 < \alpha < 1$ , the strictly increasing sequences  $\{\lambda_n\}$  and  $\{g(n)\}$ , of nonnegative terms, tending to infinity with  $n$ , satisfy the following conditions:

$$(2.1) \quad \log(\Pi/t) = O\{g(k/t)\}; \quad \text{as } t \rightarrow 0,$$

$$(2.2) \quad \{\lambda_n/(n+1)\} \uparrow \quad \text{with } n \geq n_0,$$

$$(2.3) \quad n^{1-\alpha} \Delta \lambda_n = O\{\lambda_{n+1}\}, \quad \text{as } n \rightarrow \infty,$$

$$(2.4) \quad \begin{cases} \text{(i)} & \{x/g(x)\} \uparrow \quad \text{with } x, \\ \text{(ii)} & x \frac{d}{dx} \left( \frac{1}{g(k/x)} \right) \uparrow \quad \text{with } x, \\ \text{(iii)} & \frac{d}{dx} \left( \frac{1}{g(k/x)} \right) \downarrow \quad \text{with } x, \end{cases}$$

$$(2.5) \quad \begin{cases} \text{(i)} & \left[ \frac{d}{dt} \left( \frac{1}{g(k/t)} \right) \right]_{t=1/n} = O\{n/g(n)\}, \\ \text{(ii)} & \sum_{n=1}^{\infty} (ng(n))^{-1} < \infty. \end{cases}$$

Then, if  $\Phi(t) \in \text{BV}(0, \pi)$  and  $\Lambda(t)g(k/t) \in \text{BV}(0, \pi)$ , the series

$$\sum_{n=1}^{\infty} A_n(x) \in [R, \lambda_n, 1],$$

where  $k$  is a suitable positive constant such that  $g(k/t) > 0$  for  $t > 0$ .

3. The object of this note is to generalize the above theorem. In what follows we prove the following:

**Theorem.** Let, for  $0 < \alpha < 1$ , the strictly increasing sequences  $\{\lambda_n\}$  and  $\{g(n)\}$ , of nonnegative terms, tending to infinity with  $n$ , satisfy the following conditions:

$$(3.1) \quad \log(\pi/t) = O\{g(k/t)\}, \quad \text{as } t \rightarrow 0,$$

$$(3.2) \quad \{\lambda_n/n^\delta\} \uparrow, \quad n \geq n_0, \quad 0 < \alpha < \delta < 1,$$

$$(3.3) \quad n^{1-\alpha} \Delta \lambda_n = O\{\lambda_{n+1}\}, \quad \text{as } n \rightarrow \infty,$$

$$(3.4) \quad \begin{cases} \text{(i)} & \{x^\beta/g(x)\} \uparrow \quad \text{with } x, \beta \geq 1, \\ \text{(ii)} & \frac{d}{dx} \left( \frac{1}{g(k/x)} \right) \downarrow \quad \text{with } x, \end{cases}$$

$$(3.5) \quad \begin{cases} \text{(i)} & \left[ \frac{d}{dt} \frac{1}{g(k/t)} \right]_{t=1/n} = O(n/g(n)), \\ \text{(ii)} & \sum_{n=1}^{\infty} \frac{n^{\beta-2}}{g(n)} < \infty. \end{cases}$$

Then, if  $\Phi(t) \in BV(0, \pi)$  and  $\Lambda(t)g(k/t) \in BV(0, \pi)$ , the series

$$\sum_{n=1}^{\infty} A_n(x) \in |R, \lambda_n, 1|,$$

where  $k$  is a suitable positive constant such that  $g(k/t) > 0$  for  $t > 0$ .

*Remarks:*

I. Condition (2.2)  $\Rightarrow$  (3.2) but the converse is not true. Thus (3.2) is a lighter condition.

II. For  $\beta=1$  we get a result which is a generalization of Theorem A.

III. Condition (2.4)(ii) of PREM CHANDRA is redundant. For he employs this condition in the proof of

$$(3.6) \quad \int_0^t \sin(n+1)u \frac{d}{du} \left( \frac{1}{g(k/u)} \right) du = O \left\{ \frac{1}{g(n+1)} \right\}$$

which can be proved in the following manner without the use of the above condition.

PROOF of (3.6): Case I:  $(n+1)^{-1} \leq t$ .

$$\begin{aligned} \int_0^t \sin(n+1)u \frac{d}{du} \left( \frac{1}{g(k/u)} \right) du &= \left( \int_0^{(n+1)^{-1}} + \int_{(n+1)^{-1}}^t \right) \sin(n+1)u \frac{d}{du} \frac{1}{g(k/u)} du = \\ &= I_1 + I_2, \text{ say.} \end{aligned}$$

Since  $|\sin(n+1)u| \leq (n+1)u$ ,

$$\begin{aligned} I_1 &\leq (n+1) \int_0^{(n+1)^{-1}} u \frac{d}{du} \left( \frac{1}{g(k/u)} \right) du \leq \\ &\leq \int_0^{(n+1)^{-1}} \frac{d}{du} \left( \frac{1}{g(k/u)} \right) du = \left[ \frac{1}{g(k/u)} \right]_0^{(n+1)^{-1}} = \frac{1}{g(k(n+1))} = O \left( \frac{1}{g(n+1)} \right). \end{aligned}$$

And, by virtue of second mean value theorem and the conditions (3.4) (ii), (3.5) (i), we have

$$\begin{aligned} I_2 &= \left\{ \frac{d}{du} \left( \frac{1}{g(k/u)} \right) \right\}_{u=(n+1)^{-1}}^{\xi} \int_{(n+1)^{-1}}^{\xi} \sin(n+1)u du = O \left\{ \frac{1}{g(n+1)} \right\}, \\ &\quad (n+1)^{-1} < \xi < t. \end{aligned}$$

Case II:  $(n+1)^{-1} > t$ . In this case we write,

$$\begin{aligned} \int_0^t \sin(n+1)u \frac{d}{du} \left( \frac{1}{g(k/u)} \right) du &= \left( \int_0^{(n+1)^{-1}} - \int_t^{(n+1)^{-1}} \right) \sin(n+1)u \left( \frac{d}{du} \frac{1}{g(k/u)} \right) du = \\ &= I_1 - I'_2, \text{ say.} \end{aligned}$$

Proceeding as in the case of  $I_1$  it can be easily shown that

$$I'_2 = O\left(\frac{1}{g(n+1)}\right).$$

This completes the proof of (3.6).

4. For the proof of our theorem we require the following lemmas:

**Lemma 1.** *If  $\{\lambda_n/n^\delta\}^\dagger$ ,  $n \geq n_0$ ,  $0 < \delta < 1$ , then*

$$K(n, t) = O\left\{\frac{\lambda_n t^{-\delta}}{n^\delta}\right\},$$

uniformly in  $0 < t \leq \pi$ .

PROOF. By virtue of hypothesis and Abel's Lemma

$$K(n, t) \leq \frac{\lambda_n}{n^\delta} \text{Max} \left| \sum_{v \geq n_0} \frac{\sin vt}{(v+1)^{1-\delta}} \right| + O(1).$$

But

$$\sum_{v=n_0}^m \frac{\sin vt}{(v+1)^{1-\delta}} = \sum_{v=n_0}^{[1/t]} \frac{\sin vt}{(v+1)^{1-\delta}} + \sum_{[1/t]+1}^m \frac{\sin vt}{(v+1)^{1-\delta}} = L_1 + L_2.$$

$$L_1 = \sum_{v=n_0}^{[1/t]} \frac{(v+1)^\delta \sin vt}{(v+1)} \leq \left(1 + \left[\frac{1}{t}\right]\right)^\delta \text{Max} \left| \sum_{v \geq n_0} \frac{\sin vt}{(v+1)} \right| = O(t^{-\delta}),$$

uniformly in  $0 < t \leq \pi$ .

$$L_2 = \sum_{v=[1/t]+1}^m \frac{\sin vt}{(v+1)^{1-\delta}} \leq t^{1-\delta} \text{Max} \left| \sum_{v \geq [1/t]} \sin vt \right| = O(t^{-\delta}),$$

uniformly in  $0 < t \leq \pi$ . Hence

$$K(n, t) = O\left(\frac{\lambda_n t^{-\delta}}{n^\delta}\right).$$

**Lemma 2.** *If for  $\beta \geq 1$ ,  $\{x^\beta/g(x)\}^\dagger$  with  $x$  and  $g(x) \uparrow \infty$ ,  $x \rightarrow \infty$ , then*

$$\int_0^t \frac{\sin(n+1)u}{ug(k/u)} du = O\{n^{\beta-1}/g(n+1)\},$$

uniformly in  $0 < t \leq \pi$ .

PROOF. Case I:  $(n+1)^{-1} \leq t$

$$\begin{aligned} \int_0^t \frac{\sin(n+1)u}{ug(k/u)} du &= \left( \int_0^{(n+1)^{-1}} + \int_{(n+1)^{-1}}^t \right) \frac{\sin(n+1)u}{ug(k/u)} du. \\ &= M_1 + M_2, \text{ say.} \end{aligned}$$

Now as shown by Prem Chandra [1, p. 337]

$$M_1 = O\left\{\frac{1}{g(n+1)}\right\}, \quad n \rightarrow \infty.$$

And, by virtue of the first hypothesis

$$\begin{aligned} M_2 &= \int_{(n+1)^{-1}}^t \frac{(k/u)^\beta}{g(k/u)} \frac{\sin(n+1)u}{k^\beta u^{1-\beta}} du = \frac{(n+1)^\beta}{g(k(n+1))} \int_{(n+1)^{-1}}^\xi u^{\beta-1} \sin(n+1)u du = \\ &\left(\frac{1}{n+1} < \xi < t\right) = \frac{(n+1)^\beta}{g(k(n+1))} \xi^{\beta-1} \int_\eta^\xi \sin(n+1)u du, = \\ &((n+1)^{-1} < \eta < \xi) = O\{n^{\beta-1}/g(n+1)\}, \end{aligned}$$

uniformly in  $0 < t \leq \pi$ .

Case II:  $(n+1)^{-1} > t$ .

$$\int_0^t \frac{\sin(n+1)u}{ug(k/u)} du = \left(\int_0^{(n+1)^{-1}} - \int_t^{(n+1)^{-1}}\right) \frac{\sin(n+1)u}{ug(k/u)} du = M_1 - M'_2.$$

Now

$$M'_2 = \int_t^{(n+1)^{-1}} \frac{\sin(n+1)u}{ug(k/u)} du \leq (n+1) \int_t^{(n+1)^{-1}} \frac{du}{g(k/u)} = O\left(\frac{1}{g(n+1)}\right),$$

uniformly in  $0 < t \leq \pi$ .

This completes the proof of Lemma 2.

PROOF OF THEOREM 5. Following Prem Chandra [1] we have

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \Lambda(t) g(k/t) \frac{t}{g(k/t)} \frac{d}{dt} \left( \frac{\sin nt}{nt} \right) dt$$

and the series  $\sum_{n=1}^\infty A_n(x) \in |R, \lambda_n, 1|$ , if

$$\sum = \sum_{n=0}^\infty \left| \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} \sum_{v=0}^n \frac{\lambda_v}{(v+1)} \int_0^t \frac{u}{g(k/u)} \frac{d}{du} \left( \frac{\sin(v+1)u}{u} \right) du \right| = O(1),$$

uniformly in  $0 < t \leq \pi$ .

Now, integrating by parts we have

$$\begin{aligned} \int_0^t \frac{u}{g(k/u)} \frac{d}{du} \left( \frac{\sin(v+1)u}{u} \right) du &= \frac{\sin(v+1)t}{g(k/t)} - \int_0^t \frac{\sin(v+1)u}{ug(k/u)} du - \\ &- \int_0^t \sin(v+1)u \frac{d}{du} \left( \frac{1}{g(k/u)} \right) du. \end{aligned}$$

Therefore,

$$\begin{aligned} \Sigma &\equiv \frac{1}{g(k/t)} \sum_{n=0}^{\infty} \left| \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} K(n, t) \right| + \sum_{n=0}^{\infty} \left| \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} \sum_{v=0}^n \frac{\lambda_v}{v+1} \int_0^t \frac{\sin(v+1)u}{ug(k/u)} du \right| + \\ &+ \sum_{n=0}^{\infty} \left| \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} \sum_{v=0}^n \frac{\lambda_v}{(v+1)} \int_0^t \sin(v+1)u \frac{d}{du} \left( \frac{1}{g(k/u)} \right) du \right| = \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.} \end{aligned}$$

Now, we write, for  $T = \left[ t^{-\frac{\delta}{\delta-\alpha}} \right]$

$$\Sigma = \sum_{n=0}^{T-1} + \sum_{n=T}^{\infty} = \Sigma_{1,1} + \Sigma_{1,2}, \text{ say.}$$

As shown by Prem Chandra [1, p. 339], we have

$$\Sigma_{1,1} = O(1),$$

uniformly in  $0 < t \leq \pi$ .

Now, using Lemma 1, we have

$$\begin{aligned} \Sigma_{1,2} &= \frac{1}{g(k/t)} \sum_{n=T}^{\infty} \left| \frac{\Delta \lambda_n}{\lambda_n \lambda_{n+1}} K(n, t) \right| = O \left\{ \frac{1}{g(k/t)} \sum_{n=T}^{\infty} \frac{|\Delta \lambda_n|}{\lambda_n \lambda_{n+1}} \cdot \frac{\lambda_n t^{-\delta}}{n^{\delta}} \right\} = \\ &= O \left\{ \frac{1}{g(k/t)} \sum_{n=T}^{\infty} \frac{\lambda_{n+1}}{n^{1-\alpha}} \cdot \frac{t^{-\delta}}{\lambda_{n+1} n^{\delta}} \right\} = O \left\{ \frac{t^{-\delta}}{g(k/t)} \sum_{n=T}^{\infty} \frac{1}{n^{1+\delta-\alpha}} \right\} = O(1), \end{aligned}$$

uniformly in  $0 < t \leq \pi$ . And, by virtue of Lemma 2 and (3.5) (ii)

$$\begin{aligned} \Sigma_2 &= O \left\{ \sum_{n=0}^{\infty} \frac{|\Delta \lambda_n|}{\lambda_n \lambda_{n+1}} \sum_{v=0}^n \lambda_v \cdot \frac{(v+1)^{\beta-2}}{g(v+1)} \right\} = \\ &= O \left\{ \sum_{v=0}^{\infty} \frac{\lambda_v (v+1)^{\beta-2}}{g(v+1)} \sum_{n=v}^{\infty} \left( \frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \right) \right\} = O \left\{ \sum_{v=0}^{\infty} \frac{(v+1)^{\beta-2}}{g(v+1)} \right\} = O(1), \end{aligned}$$

uniformly in  $0 < t \leq \pi$ . Also, by using (3.6), we have  $\Sigma_3 = O(1)$ , uniformly in  $0 < t \leq \pi$ .

This completes the proof of the theorem.

I wish to thank Dr. S. M. MAZHAR for his valuable suggestions and guidance which I received from him during the preparation of this paper.

### References

- [1] P. CHANDRA, Absolute summability by Riesz means, *Pacific J. Math.* **34** (1970), 335—341.

A. M. U., Aligarh, India  
(Received September 12, 1972.)