A new proof on the free product structure of Hecke groups

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In this paper, matrices are viewed as linear fractional transformations on $H = \{\tau\colon \operatorname{Im} \tau > 0\}$. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, and $A = A_\lambda = TS_\lambda$. Let Z denote the integers and let C denote the set of λ of the form $\lambda = 2\cos(\pi/q)$, where $q \in Z$, $q \geqq 3$. It is well-known [2, p. 1399] that when $\lambda \in C$, the Hecke group $G(\lambda) = \langle A, T\rangle$ is the free product of $\langle A\rangle$ and $\langle T\rangle$ (write $G(\lambda) = \langle A\rangle * \langle T\rangle$). We present here a short new proof of this theorem. A short proof in the case $\lambda = 1$ is given in [1].

Lemma. *Suppose that* $\lambda \in C$, $\tau = x + iy \in H$, *and* $I \neq W \in \langle A\rangle$. *If* $x \geqq 0$, *then* $\operatorname{Re}(W\tau) < 0$.

PROOF. Write $\lambda = 2\cos\alpha$ where $\alpha = \pi/q$. It is easily proved by induction that for all $n \in Z$,

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \csc\alpha \begin{pmatrix} \sin\alpha(1-n) & -\sin\alpha n \\ \sin\alpha n & \sin\alpha(n+1) \end{pmatrix}.$$

Since A has order q, we may write $W = A^m$, where $1 \leqq m \leqq q-1$. Write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}.$$

Then

$$\operatorname{Re}(W\tau) = \frac{(ax+b)(cx+d) + acy^2}{|c\tau + d|^2}.$$

Note that $a, b \leqq 0$ and $c, d \geqq 0$. Hence, if $x \geqq 0$, then $acy^2 \leqq 0$ and $(ax+b)(cx+d) \leqq 0$. Moreover, one of these two inequalities is strict; for, if $(ax+b)(cx+d) = 0$, then $bd = 0$ so that $m = q-1$ and hence $acy^2 < 0$. Thus $\operatorname{Re}(W\tau) < 0$. QED.

Theorem. If $\lambda \in C$, then $G(\lambda) = \langle A\rangle * \langle T\rangle$.

PROOF. Let $\lambda \in C$. Suppose that $I \neq W_j \in \langle A\rangle$ $(j = 1, 2, \ldots)$. By the Lemma, $\operatorname{Re} W_1(i) < 0$, so that $\operatorname{Re} TW_1(i) > 0$. It follows similarly by induction that for all $n \geqq 1$, $\operatorname{Re} TW_n \ldots TW_1(i) > 0$. It follows that I and T are the only reduced words in $\langle A, T\rangle$ which fix i, and hence $G(\lambda) = \langle A\rangle * \langle T\rangle$.

References

[1] D. HARDY and R. WISNER, New proof of a basic theorem about the modular group, *Publ. Math. (Debrecen)* **18** (1971), 109—110.
[2] R. C. LYNDON and J. L. ULLMAN, Groups generated by two parabolic linear fractional transformations, *Can. J. Math.* **21** (1969), 1388—1403.

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