

The integrability class of the sine transform of a monotonic function

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1. A non-decreasing, continuous and real-valued function Φ defined on the non-negative half line and vanishing only at the origin is called an Orlicz function (OF). Function $\Phi \in OF$ is said to satisfy Δ_2 condition for large u if there are constants $C > 0$ and $u_0 \geq 0$ such that $\Phi(2u) \leq C\Phi(u)$, $u \geq u_0$.

2. Recently BOAS [3] has proved the following theorem for Fourier transform by a method which is rather more direct than those that have been used for similar problems about Fourier series. His method depends on the Steffensen's version of Jensen's inequality (see MITRINOVIĆ [5], p. 109) and a theorem of EDMOND'S [4] on Parseval's theorem for monotonic functions.

Theorem A. If $f(x) \downarrow 0$, $x^{1/p}f(x) \in L^p(0, 1)$ and

$$F(x) = \sqrt{\left(\frac{\pi}{2}\right)} \int_0^\infty f(t) \sin xt \, dt,$$

then $x^{r+1-2/p}F(x) \in L^p(0, \infty)$ provided that $x^{-r}f(x) \in L^p(0, \infty)$, where $p > 1$ and $-1/p < r < 1/p$.

It may be remarked that in Theorem A, the condition $x^{1/p}f(x) \in L^p(0, 1)$ need not be mentioned because it is already implied by the condition $x^{-r}f(x) \in L^p(0, \infty)$, $-1/p < r < 1/p$.

In this note it is proposed to obtain a generalization of the above theorem. Instead of considering L^p class we would employ a more general class, namely L_Φ .

3. We prove the following theorem:

Theorem. Let $F(x)$ be the sine transform of $f(x)$. If $f(x) \downarrow 0$, $x^{-\alpha}\Phi(f(x)) \in L(0, \infty)$ and $-1 < \alpha < 1$, then $x^{\alpha-2}\Phi(xF(x)) \in L(0, \infty)$, where $\Phi(x)$ is a convex Orlicz function satisfying Δ_2 condition.

It may be observed that for $\Phi(t) = t^p$, $p > 1$, we get Theorem A.

4. We require the following lemmas for the proof of our theorems.

Lemma 1 ([2]). Let λ be a function of bounded variation on every finite sub interval of $(0, \infty)$; $\lambda(0) \leq \lambda(x)$ for all $x > 0$; and $\lambda(0) < \Lambda = \sup \lambda(x)$. Let $f(x)$ decrease and

$f(x) \geq 0$. If ψ is continuous and convex over $(0, f(0))$, $\psi(0) \leq 0$ and $\int_0^\infty d\mu(x) \leq \Lambda - \lambda(0)$, then

$$\psi \left\{ \frac{\int_0^\infty f(x) d\lambda(x)}{\int_0^\infty d\mu(x)} \right\} \leq \frac{\int_0^\infty \psi(f(x)) d\lambda(x)}{\int_0^\infty d\mu(x)}.$$

Lemma 2 ([1], p. 58). If g and B decrease to 0 on $(0, \infty)$ and $xg(x), xB(x) \in L(0, 1)$, then $B(y)b(y) \in L(0, \infty)$ iff $g(u)G(u) \in L(0, \infty)$ and Parseval's formula

$$\int_0^\infty B(y)b(y) dy = \int_0^\infty G(u)g(u) du$$

holds, G and b being the sine transforms of B and g respectively.

5. PROOF OF THE THEOREM. Taking $\lambda(x) = 1 - \cos x$, $\Lambda = 2$ and using Lemma 1, we have

$$(5.1) \quad \Phi \left(\frac{1}{2} \int_0^\infty f(x) \sin x dx \right) \leq \frac{1}{2} \int_0^\infty \Phi(f(x)) \sin x dx.$$

Since sine transform of a positive decreasing function is positive, it follows that right-hand side is positive. Also in view of the hypothesis, it is finite. Now replacing $f(x)$ by $f(xt)$, multiplying (5.1) by $t^{-\alpha}$ and integrating over $(0, \infty)$ we have

$$(5.2) \quad \int_0^\infty t^{-\alpha} \Phi \left(\frac{1}{2} \int_0^\infty f(xt) \sin x dx \right) dt \leq \frac{1}{2} \int_0^\infty t^{-\alpha} dt \int_0^\infty \Phi(f(xt)) \sin x dx.$$

Putting $t = 1/y$ and $x = yu$ in (5.2) we have

$$\int_0^\infty y^{\alpha-2} \Phi \left(\frac{1}{2} \int_0^\infty f(u) \sin yuy du \right) dy \leq \frac{1}{2} \int_0^\infty y^{\alpha-2} dy \int_0^\infty \Phi(f(u)) \sin yuy du.$$

That is to say

$$\int_0^\infty y^{\alpha-2} \Phi \left(\frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)} yF(y) \right) dy \leq \frac{1}{2} \int_0^\infty y^{\alpha-1} dy \int_0^\infty \Phi(f(u)) \sin yu du.$$

Thus it follows that

$$\begin{aligned} \int_0^\infty y^{\alpha-2} \Phi(yF(y)) dy &\leq C \int_0^\infty y^{\alpha-2} \Phi \left(\frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)} yF(y) \right) dy \\ &\leq C \int_0^\infty y^{\alpha-1} dy \int_0^\infty \Phi(f(u)) \sin yu du \\ &= C \int_0^\infty B(y)b(y) dy, \end{aligned}$$

where $B(y) = y^{\alpha-1}$, $-1 < \alpha < 1$, $g(u) = \Phi(f(u))$ and G and b are the sine transforms of B and g respectively.

Now,

$$\begin{aligned} \int_0^\infty g(u)G(u)du &= \int_0^\infty \Phi(f(u))du \int_0^\infty y^{\alpha-1} \sin yu du \\ &= \int_0^\infty u^{-\alpha} \Phi(f(u)) du \int_0^\infty t^{\alpha-1} \sin t dt *) \\ &= \Gamma(\alpha) \sin \frac{\pi\alpha}{2} \int_0^\infty u^{-\alpha} \Phi(f(u)) du \\ &< \infty, \end{aligned}$$

by virtue of the hypothesis. Thus $Gg \in L(0, \infty)$. In view of Lemma 2, Parseval's formula holds and therefore

$$\int_0^\infty y^{\alpha-2} \Phi(yF(y)) dy < \infty.$$

Thus our theorem is proved.

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References

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*) When $\alpha = 0$, the integral $\int_0^\infty \frac{\sin t}{t} dt = \pi/2$.