

On conformally flat generalised 2-recurrent spaces

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Introduction. A non-flat Riemannian space whose curvature tensor R_{hijk} satisfies the relation

$$(1) \quad \nabla_m \nabla_l R_{hijk} = \beta_m \nabla_l R_{hijk} + a_{lm} R_{hijk}$$

where β_m and a_{lm} are not both zero and ∇ denotes the operator of covariant differentiation with respect to the metric of the space has been called elsewhere [1] a generalised 2-recurrent space and an n -space of this kind has been denoted by $G(^2K_n)$. If $\beta_m = 0$, the space reduces to what is known to be a 2-recurrent space as named by A. LICHNEROWICZ [2]. Considering conformally flat 2-recurrent spaces ROY CHOWDHURY [3] showed that the scalar curvature of such a space is necessarily zero. Later, A. H. THOMPSON [4] proved that all n -dimensional ($n \geq 3$) conformally flat 2-recurrent spaces with indefinite metric are recurrent and that every such space with definite metric is flat. In the present paper we consider conformally flat generalised 2-recurrent spaces with symmetric tensors of recurrence. In such a space of zero scalar curvature, canonical expressions have been given for the Ricci tensor and the recurrence tensor in terms of a real null vector field and a non-zero scalar field. The question whether such a space can have definite metric has been answered and the nature of the space in case of indefinite metric has been determined.

1. Conformally flat $G(^2K_n)$ ($n \geq 3$) with symmetric recurrence tensor. Let us suppose that a $G(^2K_n)$ with symmetric tensor of recurrence a_{lm} is conformally flat. Then its curvature tensor is of the form

$$(1.1) \quad R_{hijk} = g_{hj} H_{ik} - g_{hk} H_{ij} + g_{ik} H_{hj} - g_{ij} H_{hk},$$

where

$$(1.2) \quad H_{ij} = -\frac{1}{n-2} \left[R_{ij} - \frac{R}{2(n-1)} g_{ij} \right]$$

and further [5]

$$(1.3) \quad \nabla_k H_{ij} = \nabla_j H_{ik}.$$

In consequence of (1) it follows from (1.2) that

$$(1.4) \quad \nabla_l \nabla_k H_{ij} = \beta_l \nabla_k H_{ij} + a_{kl} H_{ij}.$$

Hence (1.3) gives

$$(1.5) \quad a_{kl}H_{ij} = a_{jl}H_{ik}.$$

Transvecting (1.5) with g^{kl} we get

$$\Theta H_{ij} = a_{jl}g^{kl}H_{ik},$$

where $\Theta = g^{kl}a_{kl}$, whence ¹⁾

$$(1.6) \quad \Theta R_{ij} = \frac{R}{2} \frac{n-2}{n-1} a_{ij} + \frac{R}{2} \frac{\Theta}{n-1} g_{ij}.$$

If $\Theta=0$ it follows from (1.6) that $R=0$ because $n>2$ and $a_{ij} \neq 0$.

If $\Theta \neq 0$, then $R \neq 0$ for $R=0$ would in this case imply $R_{ij}=0$ which makes the space flat, contrary to assumption. We can therefore state the following theorem.

Theorem 1. *In a conformally flat $G(2K_n)$ with symmetric recurrence tensor, the scalar curvature and the scalar Θ are either both zero or both different from zero,*

2. *Conformally flat $G(2K_n)$ with symmetric recurrence tensor and zero scalar curvature.*

In this case we suppose $R_{ij} \neq 0$ and $R=0$. Hence there exists a real vector field λ^i such that

$$R_{ij}\lambda^i\lambda^j = e \quad (e = \pm 1).$$

Let $\mu_i \stackrel{\text{def}}{=} R_{ij}\lambda^j$. Then we have $\lambda^i\mu_i = e$.

¹⁾ Proof of (1.6): From the Bianchi identity we have

$$\nabla_m \nabla_l R_{hijk} + \nabla_m \nabla_j R_{hikl} + \nabla_m \nabla_k R_{hilt} = 0$$

$$\text{or, } a_{lm}R_{hijk} + a_{jm}R_{hikl} + a_{km}R_{hilt} + \beta_m(\nabla_l R_{hijk} + \nabla_j R_{hikl} + \nabla_k R_{hilt}) = 0$$

$$\text{or, } a_{lm}R_{hijk} + a_{jm}R_{hikl} + a_{km}R_{hilt} = 0 \quad (\text{using Bianchi's identity})$$

$$\text{or, } a_{lm}R_{ij\kappa}^t + a_{jm}R_{i\kappa l}^t + a_{km}R_{ilt}^t = 0.$$

Contracting t and l we get

$$a_{tm}R_{ij\kappa}^t = a_{km}R_{ij} - a_{jm}R_{i\kappa}.$$

Transvecting with g^{ij} we get $a_{tm}R_{\kappa}^t = a_{km}R - a_{jm}R_{\kappa}^j$ whence

$$(*) \quad a_{jm}R_{\kappa}^j = \frac{1}{2}Ra_{km}.$$

Now from $\Theta H_{ij} = a_{jl}g^{kl}H_{ik}$ we get using (1.2)

$$-\frac{1}{n-2}\Theta \left[R_{ij} - \frac{R}{2(n-1)}g_{ij} \right] = -\frac{1}{n-2}a_{jl}g^{kl} \left[R_{i\kappa} - \frac{R}{2(n-1)}g_{i\kappa} \right]$$

or,

$$\Theta R_{ij} - \frac{R\Theta}{2(n-1)}g_{ij} = a_{jl}R_i^l - \frac{R}{2(n-1)}a_{ji} = a_{ij}R_i^l - \frac{R}{2(n-1)}a_{ij} \quad (a_{ij} \text{ is symmetric})$$

$$= \frac{1}{2}Ra_{ij} - \frac{R}{2(n-1)}a_{ij} \quad (\text{by } (*))$$

$$= \frac{R}{2} \frac{n-2}{n-1} a_{ij}.$$

In consequence of (1.5) we have

$$R_{ij}a_{kl} = R_{ik}a_{jl}.$$

So

$$(2.1) \quad R_{ij}a_{kl}\lambda^k = \mu_i a_{jl}.$$

Contraction with λ^i gives

$$(2.2) \quad ea_{jl} = \mu_j a_{kl}\lambda^k$$

Let

$$\varrho \stackrel{\text{def}}{=} a_{ij}\lambda^i\lambda^j.$$

Then

$$\mu_j\varrho = ea_{jl}\lambda^l,$$

i. e.:

$$e\mu_l = a_{kl}\lambda^k.$$

Thus we have²⁾ from (2.2)

$$(2.3) \quad a_{jl} = \varrho\mu_j\mu_l.$$

Since $a_{jl} \neq 0$, $\varrho \neq 0$. From (2.1) we get

$$R_{ij}\varrho\mu_k\mu_l\lambda^k = \mu_i a_{jl}.$$

Whence

$$(2.4) \quad R_{ij} = e\mu_i\mu_j.$$

Since $R=0$, it follows from (2.4) that μ_i is a null vector field.

If the metric is definite it cannot contain a real null vector field. Hence the assumption that $R_{ij} \neq 0$ is false. We can therefore state the following theorems:

Theorem 2. *In a conformally flat $G(2K_n)$ with symmetric a_{im} and zero scalar curvature, there exist a real null vector field μ_i and a non-zero scalar field ϱ such that the Ricci tensor R_{ij} and the recurrence tensor a_{ij} have the canonical forms:*

$$R_{ij} = e\mu_i\mu_j (e = \pm 1) \quad \text{and} \quad a_{ij} = \varrho\mu_i\mu_j.$$

Theorem 3. *A conformally flat $G(2K_n)$ with symmetric recurrence tensor and zero scalar curvature cannot admit a definite metric.*

Since $\nabla_K R_{ij} = \nabla_j R_{iK}$ and $R_{ij} = e\mu_i\mu_j$,

we get

$$(2.5) \quad \mu_i(\nabla_K \mu_j - \nabla_j \mu_K) + (\nabla_K \mu_i)\mu_j - (\nabla_j \mu_i)\mu_K = 0.$$

Let $t_K = \lambda^i \nabla_K \mu_i$. Then transvecting (2.5) with $e\lambda^i \lambda^K$ we get

$$(2.6) \quad \lambda^K \nabla_K \mu_j = 2t_j - e\sigma\mu_j$$

where $\sigma = \lambda^m t_m = \lambda^m \lambda^n \nabla_m \mu_n$.

²⁾ Proof of (2.3): Since $ea_{jl} = \mu_j a_{kl}\lambda^k$, $ea_{jl}\lambda^l = \mu_j a_{kl}\lambda^k\lambda^l$ or, $ea_{jl}\lambda^l = \mu_j\varrho$, $a_{jl}\lambda^l = e\mu_j\varrho$. Hence $ea_{jl} = \mu_j e\mu_l\varrho = e\mu_j\mu_l$.

Again from (2.5)

$$(2.7) \quad \nabla_K \mu_i = e(\mu_i t_K + 2\mu_K t_i) - 2\sigma\mu_i\mu_K.$$

Put

$$p_i = t_i - e\sigma\mu_i.$$

Then (2.7) can be expressed as

$$(2.8) \quad \nabla_K \mu_i = e(\mu_i t_K + 2p_i\mu_K).$$

Again from $\nabla_i \nabla_K R_{ij} = \beta_i \nabla_K R_{ij} + a_{Kl} R_{ij}$, we get

$$(2.9) \quad \begin{aligned} & \beta_i(\mu_j \nabla_K \mu_i + \mu_i \nabla_K \mu_j) + \varrho\mu_i\mu_j\mu_K\mu_l = \\ & = \mu_j \nabla_i \nabla_K \mu_i + \mu_i \nabla_i \nabla_K \mu_j + \nabla_K \mu_i \nabla_l \mu_j + \nabla_l \mu_i \nabla_K \mu_j. \end{aligned}$$

Transvecting (2.9) with λ^i and λ^j we get

$$(2.10) \quad \beta_i(t_K \mu_j + e \nabla_K \mu_j) + e \varrho \mu_j \mu_K \mu_l = \mu_j \lambda^i \nabla_i \nabla_K \mu_i + e \nabla_i \nabla_K \mu_j + t_K \nabla_l \mu_j + t_l \nabla_K \mu_j$$

and

$$(2.11) \quad 2e\beta_i t_K + \varrho\mu_K \mu_l - 2t_K t_l = 2e\lambda^j \nabla_l \nabla_K \mu_j.$$

Using (2.11) we get from (2.10)

$$(2.12) \quad e \nabla_l \nabla_K \mu_j = e\beta_l \nabla_K \mu_j + \frac{1}{2} e \varrho \mu_j \mu_K \mu_l + e \mu_j t_K t_l - t_K \nabla_l \mu_j - t_l \nabla_K \mu_j.$$

Using (2.12) and (2.8) we have from (2.9) by straightforward calculations

$$(2.13) \quad 8p_i p_j \mu_K \mu_l = 0.$$

Hence $p_i = 0$, i. e.,

$$t_i = e\sigma\mu_i.$$

Therefore $\nabla_K \mu_i = \sigma\mu_i\mu_K$. From $R_{ij} = e\mu_i\mu_j$ we get $\nabla_K R_{ij} = \varphi_K R_{ij}$, where $\varphi_K = 2\sigma\mu_K$. From this we get

$$(2.14) \quad \nabla_l R_{hijk} = \varphi_l R_{hijk}.$$

If $\varphi_l = 0$, the space becomes flat which is impossible by assumption. Hence $\varphi_l \neq 0$. This means that $\sigma \neq 0$. Thus incidentally we see that λ^i and μ^i introduced in this section must not satisfy the equation $\lambda^i \lambda^j \nabla_i \mu_j = 0$.

We now state the following theorem:

Theorem 4. *If there exists a conformally flat $G(2K_n)$ with symmetric recurrence tensor and zero scalar curvature then the space is a recurrent space.*

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