

On Subobjects, Quotients, Kernels, Cokernels in a Partially Ordered Category

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1. Introduction. Preliminaries. Following MAC LANE [1] we call a category $\mathcal{B}$ a partially ordered category, if each set $\text{Rel}(A, B)$ of morphisms $f: A \rightarrow B$ satisfies the following requirements:

(I-a) To each $f: A \rightarrow B$ there is a unique $f^*: B \rightarrow A$ with

$$f^{**} = f = ff^*f, \quad (fg)^* = g^*f^*.$$

(I-b) Each $\text{Rel}(A, B)$ is a modular lattice under a partial order relation " $\subset$ " such that for $g, f: A \rightarrow B$, $g \subset f$ implies $g^* \subset f^*$, $gh \subset fh$.

(II-a) $hh^* \subset f^*f \cup 1$ implies $(f \cap g)h \supset fh \cap gh$.

(II-b) $hh^* \supset f^*f \cap 1$ implies $(f \cup g)h \subset fh \cup gh$.

(II-c) $f, g \in \text{Rel}(A, B)$ implies $f^*g \cap 1_A \subset f^*f \subset f^*g \cup 1_A$.

(II-d) $g \subset f$, $g^*g \cap 1 = f^*f \cap 1$, $gg^* \cup 1 = ff^* \cup 1$ implies $g = f$.

(III-a) For each pair of objects A, B there exist $N(A, B), P(A, B)$ in $\text{Rel}(A, B)$ such that $f \in \text{Rel}(A, B)$ implies $N \subset f \subset P$.

(III-b) $N(C, B), P(A, C) = N(D, B)P(A, D)$, $NN = N$, $PP = P$.

(III-c) $NPN = N$, $PNP = P$.

These axioms are self-dual in the lattice sense.

They are valid in the standard model $\mathcal{M}$, which is the category of all (left) modules $A, B, \dots$ over a fixed ring and of morphisms all submodules of the direct sum $A \oplus B$ (these morphisms are called "aditive relations" in [1] and [2], "Korrespondenzen" in [5], "Correspondences" in [9], "homomorphic relations" in [8], "linear relations" in [4]).

In [5] D. PUPPE has defined a relation from the object A to the object B of an abelian category $\mathcal{A}$ as a subobject of $A \oplus B$ and has given (§ 2 in [5]) a construction of the category $\mathcal{K}(\mathcal{A})$ of relations over $\mathcal{A}$; $\mathcal{K}(\mathcal{A})$ satisfies all axioms of S. Mac Lane (§ 6.10 and § 9 in [5]).

We mention that, from a different point of view, P. HILTON has described the construction of the category of relations based on an abelian category by means of a fractional calculus ([9] and [10]).

A partially ordered category $\mathcal{B}$ is a particular "category with involution"; categories with involution have been firstly defined by Puppe ([5], § 1.3) and have been called "categories of relations" by H.-B. BRINKMANN ([6] and [7]). Indeed, if in

the first group of axioms given by Mac Lane we cancel the equality $f=ff^*f$ and we weaken the assertion that each $\text{Rel}(A, B)$ is a modular lattice by assuming merely that $\text{Rel}(A, B)$ is partially ordered, then we obtain just the definition of a category with involution.

The axioms I, II, III from the definition of a partially ordered category $\mathcal{B}$ suffice to prove a number of basic properties valid for homomorphic relations or relations in an abelian category, but do not characterize the relations in an abelian category: in [5] "Beispiel A" from § 7 is a category with involution $\mathcal{K}$ which satisfies axioms I, II, III, but one could never endow the subcategory of "maps" ("eigentliche Morphismen" corresponding to usual maps or homomorphisms, defined by the conditions $f^*f \supset 1$, $ff^* \subset 1$) with a structure of pre-additive category!).

Finding all this, D. PUPPE has imposed ([5]) a set of other axioms on a category with involution which are sufficient conditions that the subcategory of "maps" be abelian and are necessarily satisfied by relations in an abelian category $\mathcal{A}$. The axioms due to D. Puppe, imply all axioms from [1] and are clearly more restrictive than the latter (we must except the condition of modularity for the lattice $\text{Rel}(A, B)$ from I-b).

Here, we take an other point of view. Without requiring any other axiom, we remain within the frame of conditions I, II, III imposed on $\mathcal{B}$ by Mac Lane and we show that $\mathcal{B}$ can be embedded in a category with involution $\overline{\mathcal{B}}$ in which every "map" of $\mathcal{B}$ has a kernel and a cokernel. Our immediate aim is to show that the seemingly peculiar definitions of subobjects, quotients, kernels in $\mathcal{B}$ (due to Mac Lane) amounts in $\overline{\mathcal{B}}$ to the usual definitions with monomorphisms, epimorphisms and, respectively, projective limits; so, we are not concerned only with the question of the existence of kernels and cokernels for maps of $\mathcal{B}$, we give in Theorems 2 and 3 a general characterization of the "kernel" and the "cokernel" ("kernel" in the sense of [1]) of an arbitrary morphism of $\mathcal{B}$ (not necessarily of a map).

It is known that in $\mathcal{B}$ the symmetric idempotents $u \in \text{Rel}(A, A)$, $u = u^* = uu$, form a sublattice which we denote by $\overline{\text{Rel}}(A, A)$; all $s \subset 1_A$ and $q \supset 1_A$ are symmetric idempotents, called subobjects and quotients of A , respectively.

We recall also that for each object A there is a lattice isomorphism between the lattice of subobjects $s \subset 1_A$ and that of quotients $q \supset 1_A$, given by $\varphi(s) = 1 \cup sO^*$, $\psi(q) = 1 \cap qO$, $\varphi^{-1} = \psi$; $O(A, B)$ is defined by $O(A, B) = N(B, B)P(A, B) = N(A, B)P(A, A)$. S. Mac Lane introduces for $f: A \rightarrow B$, $\text{Def } f = f^*f \cap 1_A$, $\text{Ker } f = \psi(f^*f \cup 1)$ and shows that $\text{Ker } f = 1 \cap f^*Nf$. We set $f^*f \cup 1 = \text{Coim } f$ and $\text{Im } f = \text{Def } f^* = ff^* \cap 1_B$ and introduce also $\text{Coker } f = \varphi(\text{Im } f)$ so that $\text{Coker } f = 1_B \cup fPf^*$ (see also [3]).

2. Definition. We define the category $\overline{\mathcal{B}}$ associated to a partially ordered category $\mathcal{B}$ by

a) $\text{Ob } \mathcal{B} \subset \text{Ob } \overline{\mathcal{B}}$; for each $u \in \overline{\text{Rel}}(A, A)$, $u = u_A \in \text{Ob } \overline{\mathcal{B}}$; the new objects are but the symmetric idempotent morphisms of $\mathcal{B}$.

b) For $A, B \in \text{Ob } \mathcal{B}$, $\text{Hom}_{\overline{\mathcal{B}}}(A, B) = \text{Rel}(A, B)$; for $B \in \text{Ob } \mathcal{B}$ and $u \in \overline{\text{Rel}}(A, A)$, $\text{Hom}_{\overline{\mathcal{B}}}(u_A, B) = \text{Hom}(u_A, B) = \{f: A \rightarrow B / fu = f\}$; for $A \in \text{Ob } \mathcal{B}$ and $v \in \overline{\text{Rel}}(B, B)$, $\text{Hom}_{\overline{\mathcal{B}}}(A, v_B) = \text{Hom}(A, v_B) = \{g: A \rightarrow B / vg = g\}$; $\text{Hom}_{\overline{\mathcal{B}}}(u_A, v_B) = \text{Hom}(u_A, v_B) = \{f: A \rightarrow B / fu = vf = f\}$.

c) The composition of morphisms in $\bar{\mathcal{B}}$ is the same as in $\mathcal{B}$; we are allowed to state this because, if $f \in \text{Rel}(A, B)$ and $g \in \text{Hom}(B, u_C)$, then $gf \in \text{Hom}(A, u_C)$ ($ug = g \Rightarrow ugf = gf$), if $f \in \text{Hom}(u_A, B)$ and $g \in \text{Rel}(B, C)$, then $gf \in \text{Hom}(u_A, C)$ ($fu = f \Rightarrow gfu = gf$), if $f \in \text{Hom}(u_A, B)$ and $g \in \text{Hom}(B, u_C)$, then $gf \in \text{Hom}(u_A, u_C)$, if $f \in \text{Hom}(A, u_B)$ and $g \in \text{Hom}(u_B, C)$, then $gf \in \text{Rel}(A, C)$.

Remarks. 1. Denote by $\text{Codef } f = \text{Coim } f^* = ff^* \cup 1_B$; $f \in \text{Hom}(u_A, B)$ if and only if $\text{Def } f \subset u \subset \text{Coim } f$; $f \in \text{Hom}(A, v_B)$ if and only if $\text{Im } f \subset v \subset \text{Codef } f$.

Indeed, according to MAC LANE [1] $fu = f$ if and only if $f^*f \cap 1_A \subset u \subset f^*f \cup 1_A$.

2. The identity morphism in $\text{Hom}(u_A, u_A)$ is just $u = u_A$ from $\text{Rel}(A, A)$.

3. One can easily check the validity of axioms I for $\bar{\mathcal{B}}$. For example, $f \in \text{Hom}(u_A, B)$ implies $fu = f$ i.e. $uf^* = f^*$ and thus $f^* \in \text{Hom}(B, u_A)$. If $f, g \in \text{Hom}(u_A, B)$, then $f \cap g \in \text{Hom}(u_A, B)$: the necessary and sufficient condition (given in [1]) for the distributive law $(f \cap g)h = fh \cap gh$, namely $hh^* \subset f^*f \cup 1_B$, can be applied in $(f \cap g)u$, as $uu^* = u \subset \text{Coim } f$; then we have $(f \cap g)u = fu \cap gu = f \cap g$. Similarly, $(f \cup g)u = fu \cup gu = f \cup g$, as $uu^* \supset f^*f \cap 1_A$.

4. If we consider for each couple $A, B \in \text{Ob } \mathcal{B}$ only those morphisms $f: A \rightarrow B$ for which we have $f^*f \supset 1_A$ and $ff^* \subset 1_B$, we obtain two subcategories $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The morphisms of $\mathcal{B}$ which satisfy simultaneously both inequalities form the subcategory $M(\mathcal{B})$ of "maps" in $\mathcal{B}$.

5. Corresponding to the above subcategories of $\mathcal{B}$, we have the subcategories $\bar{\mathcal{B}}_1$, and $\bar{\mathcal{B}}_2$ and $M(\bar{\mathcal{B}})$ of $\bar{\mathcal{B}}$:

$$\text{Hom}_{\bar{\mathcal{B}}_1}(A, B) = \text{Hom}_{\mathcal{B}_1}(A, B), \quad \text{Hom}_{\bar{\mathcal{B}}_2}(A, B) = \text{Hom}_{\mathcal{B}_2}(A, B),$$

$$\text{Hom}_{\bar{\mathcal{B}}_1}(u_A, B) = \{f: u_A \rightarrow B, f^*f \supset 1_{u_A} = u\},$$

$$\text{Hom}_{\bar{\mathcal{B}}_1}(A, v_B) = \{g: A \rightarrow v_B, g^*g \supset 1_A\},$$

$$\text{Hom}_{\bar{\mathcal{B}}_1}(u_A, v_B) = \{f: u_A \rightarrow v_B, f^*f \supset u\},$$

$$\text{Hom}_{\bar{\mathcal{B}}_2}(u_A, B) = \{f: u_A \rightarrow B, ff^* \subset 1_B\},$$

$$\text{Hom}_{\bar{\mathcal{B}}_2}(A, v_B) = \{g: A \rightarrow v_B, gg^* \subset v\},$$

$$\text{Hom}_{\bar{\mathcal{B}}_2}(u_A, v_B) = \{g: u_A \rightarrow v_B, gg^* \subset v\}.$$

Clearly $\mathcal{B}$ is a full subcategory of $\bar{\mathcal{B}}$ and $M(\mathcal{B})$ is a full subcategory of $M(\bar{\mathcal{B}})$.

3. Theorems.

Theorem 1. *If $s \subset 1_A$ is a subobject of $A \in \text{Ob } \mathcal{B}$, then the morphism $s \in \text{Hom}(s_A, A)$ is a monomorphism in $\bar{\mathcal{B}}$. If $q \supset 1_A$ is a quotient of $A \in \text{Ob } \mathcal{B}$, then the morphism $q \in \text{Hom}(A, q_A)$ is an epimorphism in $\bar{\mathcal{B}}$.*

PROOF. It is clear that $s \in \text{Hom}(s_A, A)$. If $sg = sh$ for $g, h \in \text{Hom}(X, s_A)$, we must have also $sg = g$ and $sh = h$, so it follows obviously $g = h$.

Similarly $q \in \text{Hom}(A, q_A)$ and from $gq = hq$, for $g, h \in \text{Hom}(q_A, X)$ it follows $g = h = gq = hq$.

For $f: A \rightarrow B$ let $\text{Ker } f = 1_A \cap f^*N(B, B)f$ be an object of the category $\bar{\mathcal{B}}$ and $i = 1_A \cap f^*N(B, B)f$ a morphism (from $\text{Ker } f$ to A) in $\bar{\mathcal{B}}$.

Theorem 2. $\text{Ker } f$ and i have the following properties:

- (a) For $e = (ff^* \cup 1_B)O(A, B)$ we have $fi = ei$.
 (a') $\text{Im } i \subset \text{Ker } f$.
 (b) For each $X \in \text{Ob } \bar{\mathcal{B}}$ and $g \in \text{Hom}(X, A)$ with $\text{Im } g \subset \text{Ker } f$ (that is $gg^* \cap 1_A \subset 1_A \cap f^*Nf$), there exists a unique morphism $h \in \text{Hom}(X, \text{Ker } f)$, such that $ih = g$.

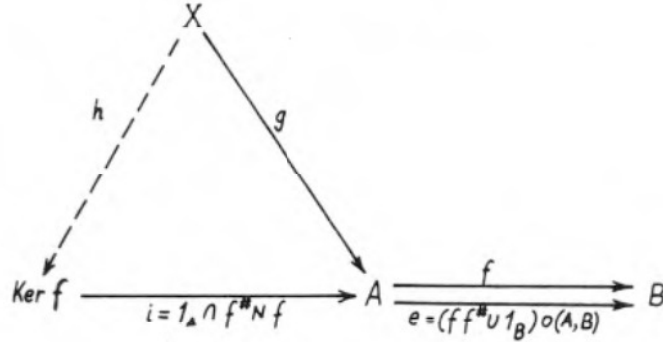


Fig. 1

(c) If $Y \in \text{Ob } \bar{\mathcal{B}}$ and $j \in \text{Hom}(Y, A)$ have the properties:

[a'] $\text{Im } j \subset \text{Ker } f$;

[b] $\forall X \in \text{Ob } \bar{\mathcal{B}}, \forall g \in \text{Hom}(X, A)$ with $\text{Im } g \subset \text{Ker } f, \exists$ a unique $h \in \text{Hom}(X, Y)$ such that $jh = g$;
 then there exists a unique isomorphism Θ such that the following diagram is commutative

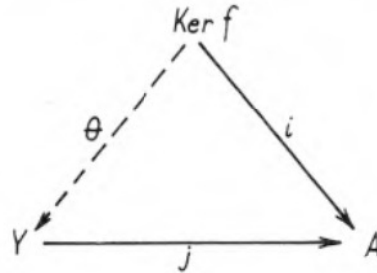


Fig. 2

(d) $\text{Im } g \subset \text{Ker } f \Rightarrow fg = eg$. In the subcategory $\bar{\mathcal{B}}_2$, $e = O(A, B)$ and the opposite implication also holds, i.e. $fg = eg \Leftrightarrow \text{Im } g \subset \text{Ker } f$ which shows (see (b)) that in $\bar{\mathcal{B}}_2$ $(\text{Ker } f, i)$ is a projective limit of the pair $(f, O(A, A))$.

PROOF. (a) Let us show firstly that $O(A, B)(1_A \cap f^*Nf) \subset f$. For $\forall f: A \rightarrow B, C \in \text{Ob } \mathcal{B}$, the following equalities can be checked by calculation:

$$\text{Ker}(N(B, C)f) = \text{Def}(N(B, C)f) = \text{Ker } f;$$

$$\text{Ker}(P(B, C)f) = \text{Def}(P(B, C)f) = \text{Def } f.$$

Hence we may write

$$\text{Def}[N(B, A)f(1_A \cap f^* Nf)] = \text{Ker}[f(1_A \cap f^* Nf)];$$

besides,

$$\text{Ker}[f(1_A \cap f^* Nf)] = (1_A \cap f^* Nf)f^* Nf(1_A \cap f^* Nf) \cap 1_A \supset$$

$$\supset (1_A \cap f^* Nf)(1_A \cap f^* Nf)(1_A \cap f^* Nf) \cap 1_A = 1_A \cap f^* Nf = \text{Def}(1_A \cap f^* Nf).$$

If $f \in \text{Rel}(A, B)$ and $g \in \text{Rel}(A, B')$, then

$$\text{Def} f \subset \text{Def} g \Rightarrow O(B, C)f \subset O(B', C)g, \quad \forall C \in \text{Ob } \mathcal{B}.$$

Indeed, $f^* f \cap 1_A \subset g^* g \cap 1_A$ implies $(f^* f \cap 1_A) \cup O^*(A, A) \subset (g^* g \cap 1_A) \cup O^*(A, A)$; constructing the dual of the equality $N(B, A)g = (g^* g \cup 1_A) \cap O(A, A)$, for $\forall g \in \text{Rel}(A, B)$, given in [1], we obtain $(f^* f \cap 1_A) \cup O^*(A, A) = P(B, A)f$; so we have $P(B, A)f \subset P(B', A)g$; it follows $N(A, C)P(B, A)f \subset N(A, C)P(B', A)g \Rightarrow O(B, C)f \subset O(B', C)g$.

Thus $\text{Def}(1_A \cap f^* Nf) \subset \text{Def}[N(B, A)f(1_A \cap f^* Nf)]$ implies

$$\begin{aligned} O(A, B)(1_A \cap f^* Nf) &\subset O(A, B)N(B, A)f(1_A \cap f^* Nf) = \\ &= N(B, B)f(1_A \cap f^* Nf) \subset 1_B f(f^* f \cap 1_A) = f. \end{aligned}$$

(The inclusion $\text{Ker} f \subset \text{Def} f$ has been proved in [1].)

A theorem due to S. MAC LANE [1] says that $g \subset h \Leftrightarrow (\text{Codef } h)g = h(\text{Def } g)$; then $O(A, B)(1_A \cap f^* Nf) \subset f$ is equivalent to

$$(ff^* \cup 1_B)O(A, B)(1_A \cap f^* Nf) = f \text{Def}[O(A, B)(1_A \cap f^* Nf)];$$

as $\text{Def}[O(A, B)(1_A \cap f^* Nf)] = \text{Ker}[P(A, B)(1_A \cap f^* Nf)] = \text{Def}(1_A \cap f^* Nf) = 1_A \cap f^* Nf$ we have just $ei = fi$.

(a') $\text{Im}(1_A \cap f^* Nf) = 1_A \cap f^* Nf = \text{Ker } f$.

(b) Since $\text{Im } g = \text{Def } g^* \subset \text{Ker } f = i \subset \text{Coim } g^*$, we have $g^* i = g^* \Leftrightarrow ig = g$, so that $g \in \text{Hom}(X, \text{Ker } f)$; we can then consider $h = g$. Suppose that $h' \in \text{Hom}(X, \text{Ker } f)$, that is $ih' = h'$, and that h' satisfies also the condition of commutativity $ih' = g$. Then we obtain necessarily $h' = g$.

(c) $\text{Im } j \subset \text{Ker } f$ implies $\text{Def } j^* \subset i \subset \text{Coim } j^*$, so that $j^* i = j^* \Leftrightarrow ij = j$; it follows that $j \in \text{Hom}(Y, \text{Ker } f)$. From $\text{Im } j \subset \text{Ker } f$ we obtain also, using (b), that there exists a unique morphism $h_1 \in \text{Hom}(Y, \text{Ker } f)$ such that $ih_1 = j$, namely just $h_1 = j$.

Applying [b] to $X = \text{Ker } f$ and $g = i$ ($\text{Im } i = \text{Ker } f$), we have that there exists a unique $h_2 \in \text{Hom}(\text{Ker } f, Y)$ with $jh_2 = i$. Let us show that this h_2 is an isomorphism.

The two conclusions obtained above for $h_1 = j$ and h_2 , imply $h_1 h_2 \in \text{Hom}(\text{Ker } f, \text{Ker } f)$, $h_2 h_1 \in \text{Hom}(Y, Y)$ and

$$i(h_1 h_2) = (ih_1)h_2 = jh_2 = i, \quad j(h_2 h_1) = (jh_2)h_1 = ih_1 = j.$$

According to (b) for $X = \text{Ker } f$ and $g = i$, there exists a unique $h_3 \in \text{Hom}(\text{Ker } f, \text{Ker } f)$ which satisfies $ih_3 = i$, take namely $h_3 = i$; by Remark 2 $i = 1_{\text{Ker } f}$, so that $h_1 h_2$ coincides with $1_{\text{Ker } f}$. In the same way, if we apply [b] to $X = Y$ and $g = j$ ($\text{Im } j \subset \text{Ker } f$), we have that there exists a unique $h_4 \in \text{Hom}(Y, Y)$ which satisfies $jh_4 = j$, precisely $h_4 = 1_Y \in \text{Hom}(Y, Y)$; it follows then $h_2 h_1 = 1_Y$.

We can now conclude that $h_2 \in \text{Hom}(\text{Ker } f, Y)$ is an isomorphism with $h_2^{-1} = h_1 = j$.

(d) $\text{Im } g \subset \text{Ker } f \Leftrightarrow gg^* \cap 1_A \subset 1_A \cap f^* Nf \Rightarrow O(A, B)(gg^* \cap 1_A) \subset O(A, B)(1_A \cap f^* Nf) \subset f$.
 $m = O(A, B)(gg^* \cap 1_A) \subset f$ implies $(\text{Codef } f)m = f(\text{Def } m)$, that is

$$(ff^* \cup 1_B)O(A, B)(gg^* \cap 1_A) = f(\text{Def } m).$$

$$\begin{aligned} \text{Def } m &= \text{Def}[O(A, B)(gg^* \cap 1_A)] = \text{Ker}[P(A, B)(gg^* \cap 1_A)] = \text{Def}(gg^* \cap 1_A) = \\ &= gg^* \cap 1_A \quad \text{and} \quad (gg^* \cap 1_A)g = (\text{Im } g)g = g. \end{aligned}$$

Then it follows immediately

$$(ff^* \cup 1_B)O(A, B)(gg^* \cap 1_A)g = f(gg^* \cap 1_A)g \Leftrightarrow eg = fg.$$

If $ff^* \subset 1_B$, $e = 1_B O(A, B) = O(A, B)$ and from $O(A, B)g = fg$ we have $O(A, B)gg^* = fgg^*$. Since $gg^* \subset 1$, $O(A, B)gg^* \subset O(A, B)$ and $fgg^* \subset f$, so that $fgg^* \subset O(A, B) \cap f = N(B, B)f$ (the last equality proved in [1]); thus $N(B, B)fgg^* \subset N(B, B)f$, which implies $\text{Ker}(fgg^*) \subset \text{Ker}(N(B, B)f)$. As $\text{Ker}(fgg^*) = \text{Ker}[O(A, B)gg^*] = \text{Def}(gg^*) = gg^*$ and $\text{Ker}(N(B, B)f) = \text{Ker } f$, we have in fact $gg^* \subset \text{Ker } f \Leftrightarrow \text{Im } g \subset \text{Ker } f$.

Thus in $\bar{\mathcal{B}}_2$, the inclusion $\text{Im } g \subset \text{Ker } f$ is equivalent to $fg = O(A, B)g$; then (a') and (b) become: $fi = O(A, B)i$ and for each $g \in \text{Hom}(X, A)$ with $fg = O(A, B)g$, there exists a unique $h \in \text{Hom}(X, \text{Ker } f)$ such that $ih = g$, which is just the necessary and sufficient condition for $(\text{Ker } f, i)$ to be a projective limit of the pair $(f, O(A, B))$ (cf. [11], ch. 2.2 and ch. 3.1).

For $f: A \rightarrow B$ we consider $\text{Coker } f = 1_B \cup fP(A, A)f^*$ as an object of $\bar{\mathcal{B}}$ and $p = 1_B \cup fP(A, A)f^*$ as a morphism from B to $\text{Coker } f$ in $\bar{\mathcal{B}}$. Then, one can prove

Theorem 3. (a) For $t = O(A, B)(f^*f \cap 1_A)$ we have $pt = pf$.

(a') $\text{Coker } f \subset \text{Coim } p$.

(b) $\forall X \in \text{Ob } \bar{\mathcal{B}}$ and $g \in \text{Hom}(B, X)$ with $\text{Coker } f \subset \text{Coim } g$ (that is $1_B \cup fP(A, A)f^* \subset g^*g \cup 1_B$), there exists a unique morphism $h \in \text{Hom}(\text{Coker } f, X)$ such that $hg = p$.

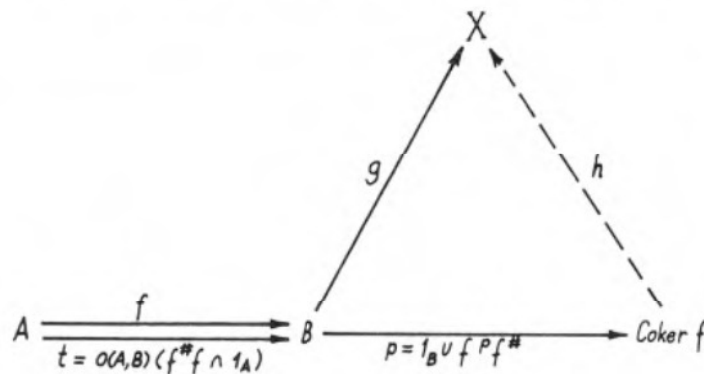


Fig. 3

(c) If $Y \in \text{Ob } \bar{\mathcal{B}}$ and $r \in \text{Hom}(B, Y)$ have the properties:

[a'] $\text{Coker } f \subset \text{Coim } r$.

[b] If $X \in \text{Ob } \bar{\mathcal{B}}$ and $g \in \text{Hom}(B, X)$ with $\text{Coker } f \subset \text{Coim } g$, $\exists$ a unique $h \in \text{Hom}(Y, X)$ such that $hr = g$;
then there exists a unique isomorphism $\theta \in \text{Hom}(Y, \text{Coker } f)$ such that $\theta r = p$

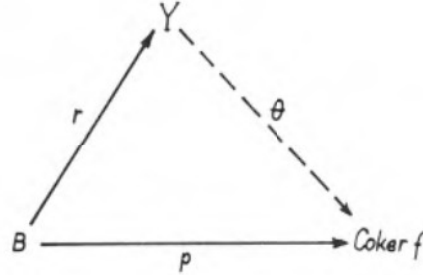


Fig. 4

(d) $\text{Coker } f \subset \text{Coim } g \Rightarrow gf = gt$. In the subcategory $\bar{\mathcal{B}}_1$, $t = O(A, B)$ and $gf = gt \Leftrightarrow \text{Coker } f \subset \text{Coim } g$, so that $(\text{Coker } f, p)$ is the inductive limit of $(f, O(A, B))$.

Corollary. For an arbitrary map $f: A \rightarrow B$ from $M(\mathcal{B})$, the pair $(f, O(A, B))$ has a kernel and a cokernel in $M(\bar{\mathcal{B}})$.

PROOF. $O(A, B)$ is a map in $\mathcal{B}$, $O(A, B) \in \text{Hom}_{M(\mathcal{B})}(A, B)$, as $O^*(A, B)O(A, B) = P(A, A) \supset 1_A$, $O(A, B)O^*(A, B) = N(B, B) \subset 1_B$. Needless to say that f and $O(A, B)$ are maps in $\mathcal{B}$, too.

According to Theorem 2, $i = 1_A \cap f^* Nf: (\text{Ker } f = i_A) \rightarrow A$ is a kernel of the pair $(f, O(A, B))$ in $\bar{\mathcal{B}}_2$; we are allowed to state the same fact in $M(\bar{\mathcal{B}})$, since not only $ii^* = i \subset 1_A$, but also $i^* i = i \supset 1_{\text{Ker } f} = i$ and thus $i \in \text{Hom}_{M(\bar{\mathcal{B}})}(\text{Ker } f, A)$ is a map in $\bar{\mathcal{B}}$.

According to Theorem 3, $p = 1_B \cap f Nf^*: A \rightarrow (\text{Coker } f = p_B)$ is a cokernel of the pair $(f, O(A, B))$ in $\bar{\mathcal{B}}_1$; we are allowed to assert the same thing in $M(\bar{\mathcal{B}})$, since $p^* p = p \supset 1_B$, but also $pp^* = p \subset 1_{\text{Coker } f} = p$ and thus $p \in \text{Hom}_{M(\bar{\mathcal{B}})}(A, \text{Coker } f)$ is a map in $\bar{\mathcal{B}}$.

Theorem 4. The category $\bar{\mathcal{M}}$ is equivalent to $\mathcal{M}$.

PROOF. We define a functor $F: \mathcal{M} \rightarrow \bar{\mathcal{M}}$ by $F(A) = A$ for a module A and $F(R) = R$ for a linear relation R (submodule in a direct sum of modules $R \subseteq A \oplus B$). It is clear that the map induced by F from $\text{Rel}(A, B)$ to $\text{Hom}_{\bar{\mathcal{M}}}(A, B) = \text{Rel}(A, B)$ is the identity mapping. Moreover, each object $X \in \text{Ob } \bar{\mathcal{M}}$ is isomorphic to an object $F(A)$, $A \in \text{Ob } \mathcal{M}$; indeed, X is either a module $A = F(A)$, or a symmetric idempotent

$R_A \subseteq A \oplus A$; in the second situation, $it^\# : \frac{\text{Def } R}{\text{Ker } R} \rightarrow R_A$ with $t : \text{Def } R \rightarrow \frac{\text{Def } R}{\text{Ker } R} \rightarrow$ canonical projection, is an isomorphism in $\mathcal{M}$: $R_A \subseteq A \oplus A$ symmetric idempotent submodule implies $R = \bigcup_{C \in \frac{\text{Def } R}{\text{Ker } R}} (C \times C)$, so that $R(it^\#) = it^\#, (ti^\#)R = ti^\#, (it^\#)(ti^\#) = R = 1_R$ and $(ti^\#)(it^\#) = 1_{\text{Def } R / \text{Ker } R}$.

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