Conditional-completion of Boolean rings by lower cuts

By ALEXANDER ABIAN (Ames, Iowa)

1. Introduction

In what follows $(B, \leqq)$ denotes a Boolean ring *not necessarily with a unit* (cf. [1, p. 130]). Hence, $(B, \leqq)$ is not necessarily a Boolean algebra. Accordingly, $(B, \leqq)$ is a distributive lattice with the least element 0 and B is closed under subtraction. This means that $(B, \leqq)$ is a partially ordered set such that for every element x and y of B, the least upper bound $x \vee y$ as well as the greatest lower bound $x \wedge y$ exists, and for every element z of B

$$z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) \tag{1}$$

which implies that for every element x, y, z of B

$$z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y). \tag{2}$$

Moreover, for every element x and y of B, the *difference* $x-y$ exists, where $d=x-y$ is the unique element of B such that

$$d \vee y = x \vee y \quad \textit{and} \quad d \wedge y = 0. \tag{3}$$

As usual, a Boolean ring $(B, \leqq)$ is called *conditionally complete* if and only if every nonempty subset S of B (which, a priori, is bounded below by 0) has a greatest lower bound *(infimum)* denoted by $\wedge S$ or by $\bigwedge_{x \in S} x$. Equivalently, a Boolean ring $(B, \leqq)$ is conditionally complete, if and only if every subset E of B which is bounded above has a least upper bound *(supremum)* denoted by $\vee E$ or by $\bigvee_{x \in E} x$. Clearly, in $(B, \leqq)$ we have $\vee \emptyset = 0$.

Obviously, a Boolean ring $(B, \leqq)$ need not be conditionally complete. However, as shown in this paper for every Boolean ring $(B, \leqq)$ there exists a conditionally complete Boolean ring $(L, \subseteq)$ (where L is the set of all the lower cuts of $(B, \leqq)$, see Definition 1 below such that there exists an isomorphism (i.e., a one-to-one mapping which preserves order in both directions) f from $(B, \leqq)$ into $(L, \subseteq)$ which is also *infima and suprema preserving*. This means that the isomorphism f is such that for every subset S of B

$$f\Big(\bigwedge_{x \in S} x\Big) = \bigwedge_{x \in S} f(x) \tag{4}$$

whenever the left side of the above equality exists. Also, for every subset E of B

$$f(\bigvee_{x\in S} x) = \bigvee_{x\in E} f(x) \tag{5}$$

whenever the left side of the above equality exists. Clearly, in (4) and (5) the left side of each equality refers to $(B, \leqq)$ whereas the right side of each equality refers to $(L, \subseteqq)$.

It is customary to refer to an isomorphism f described above or to the Boolean ring $(L, \subseteqq)$ described above, as a *conditional-completion* of the Boolean ring $(B, \leqq)$.

As shown below, the conditional-completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ that we consider in this paper, has the following special features. The lattice-theoretic order in $(L, \subseteqq)$ is the set-theoretic inclusion $\subseteqq$. Moreover, for every element X and Y of $(L, \subseteqq)$, the infimum $X \wedge Y$ is the set-theoretical intersection $X \cap Y$. However, for every element X and Y of $(L, \subseteqq)$, the supremum $X \vee Y$ is generally larger (w.r.t. $\subseteqq$) than the set-theoretical union $X \cup Y$. Also, for every element X and Y of $(L, \subseteqq)$, the difference $X - Y$ is generally smaller (w.r.t. $\subseteqq$) than the set-theoretical difference $X \setminus Y$. In fact, we show below that the conditional-completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ is a conditionally-complete Boolean ring in which

$$\wedge G = \cap G \quad \textit{for every nonempty subset } G \textit{ of } L \tag{6}$$

whereas, as mentioned, $\vee H \supseteqq \cup H$ for every subset H of L which is bounded above and $X - Y \subseteqq X \setminus Y$ for every element X and Y of $(L, \subseteqq)$.

Our conditional-completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ resembles Dedekind's conditional-completion of rational numbers by real numbers, whereby every real number r is identified with the set of all rational numbers x such that $x \leqq r$.

Also, our conditional-completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ closely resembles MacNeille's original work [2] on completion by cuts of a Boolean algebra (i.e., a Boolean ring with unit). However, as the reader will note, between [2] and the present paper, there exist substantial differences in definitions, in methods and in proofs.

Conditional-completion of partially ordered sets and in particular conditional-completion of Boolean rings are of significant importance in mathetics. However, it seems that there is nothing in print which treats these subjects elegantly, lucidly and in a self-contained readable and understandable way without resorting to complicated topological considerations. It is hoped that the present paper will contribute to the subject of the conditional-completion of Boolean rings. The obvious relation of a completion of a partially ordered set (or a Boolean ring) to the conditional-completion of the partially ordered set (or the Boolean ring) is mentioned in the sequel.

2. Lower cuts and conditional-completion of partially ordered sets

Let $(P, \leqq)$ be a partially ordered set. For every element x of P we call the subset $I(x)$ of P a *lower segment* of P if and only if

$$I(x) = \{y \mid y \in P \textit{ and } y \leqq x\}. \tag{7}$$

Based on (7) we introduce:

Definition 1. *A subset W of a partially ordered set $(P, \leqq)$ is called a lower cut of P if and only if for some nonempty subset S of P*

$$W = \bigcap_{x \in S} I(x). \tag{8}$$

In other words, a lower cut of a partially ordered set P is the intersection of a nonempty family of lower segments of P.

Let $(P, \leqq)$ be a partially ordered set, then

$$\bigcap_{x \in P} I(x) = \{0\} \quad or \quad \bigcap_{x \in P} I(x) = \emptyset \tag{9}$$

where 0 is the minimum (the least) element of P. This is because if $\bigcap_{x \in P} I(x) \neq \emptyset$ then there exists an element $h \in \bigcap_{x \in P} I(x)$ and therefore by (7) we have $h \leqq x$ for every $x \in P$. Thus, h is the (unique) minimum element of P.

Lemma 1. *Let $(P, \leqq)$ be a partially ordered set. Every nonempty subst of P which is bounded below has an infimum if and only if every nonempty subset of P which is bounded above has a supremum.*

PROOF. The proof of the lemma follows easily from the fact that for every subset S of $(P, \leqq)$ we have:

$$\inf S = \sup \{x \mid x \textit{ is a lower bound of } S\}$$

and

$$\sup S = \inf \{x \mid x \textit{ is an upper bound of } S\}.$$

Definition 2. *A partially ordered set P is called conditionally complete if and only if every nonempty subset of P which is bounded below has a greatest lower bound (or every nonempty subset of P which is bounded above has a least upper bound).*

Based on Definitions 1 and 2 we prove:

Lemma 2. *Let L be the set of all lower cuts of a partially ordred set $(P, \leqq)$. Then $(L, \subseteqq)$ is a conditionally complete partially ordered set such that for every nonempty subset G of L which is bounded below*

$$\inf G = \wedge G = \cap G. \tag{10}$$

Moreover, for every nonempty subset H of L which is bounded above

$$\sup H = \vee H \supseteqq \cup H. \tag{11}$$

PROOF. Clearly, $(L, \subseteqq)$ is a partially ordered set since $\subseteqq$ is the set-theoretical inclusion. In order to prove that $(L, \subseteqq)$ is conditionally complete, in view of Lemma 1, it is sufficient to establish (10). Now, let G be as described in the statement of the Lemma. But then by (8) we see that for some nonempty subset K of the powerset (i.e., the set of all subset) of P

$$G = \{W \mid W = \bigcap_{x \in S} I(x) \quad and \quad \emptyset \neq S \in K\}.$$

But then

$$\cap G = \bigcap_{S \in K} \Big(\bigcap_{x \in S} I(x)\Big) = \bigcap_{x \in \cup K} I(x).$$

Thus, by (8), we see that $\cap G$ is a lower cut of $(P, \leqq)$ and therefore $\cap G \in L$. Hence, from Lemma 1 and Definition 2 it follows that $(L, \subseteqq)$ is conditionally complete. Moreover, since $\subseteqq$ is the set-theoretical inclusion, in view of Lemma 1 we see that (10) implies (11).

Theorem 1. *Let $(P, \leqq)$ be a partially ordered set and $(L, \subseteqq)$ be the conditionally complete partially ordered set of all lower cuts of P. Then the function f given by*

$$f(x) = I(x) \tag{12}$$

is a one-to-one suprema and infima preserving mapping from $(P, \leqq)$ into $(L, \subseteqq)$.

PROOF. Let x and y be distinct elements of P. Without loss of generality, we may assume $x \not\leqq y$. But then by (7) and (12) we see that $x \in f(x)$ and $x \notin f(y)$. Hence $f(x) \neq f(y)$ and consequently, f is one-to-one.

Next, we show that f is order preserving, i.e., for every element x and y of P, we have:

$$x < y \textit{ if and only if } f(x) \subset f(y). \tag{13}$$

Let $x < y$ and let $z \in f(x)$. By (7) and (12) we see that $z \leqq x < y$ and therefore $z \in f(y)$ which implies $f(x) \subset f(y)$. Conversely, let $f(x) \subset f(y)$. But then clearly by (7) and (12) we have $x < y$.

Now, we show that f is infima preserving. In view of (13), it is enough to show that f preserves the infimum of every nonempty subset of P. Let S be a nonempty subset of P. By (12) we have:

$$f[S] = \{I(x) \mid x \in S\}.$$

On the other hand, from (10) it follows that

$$\inf f[S] = \wedge f[S] = \cap f[S] = \bigcap_{x \in S} I(x).$$

Let $s = \inf S = \wedge S$. But then, by (7) and (12) we have

$$f(s) = I(s) \subseteqq \bigcap_{x \in S} I(x).$$

On the other hand, if $y \in \bigcap_{x \in S} I(x)$ then clearly $y \leqq s$ since $s = \inf S$. Hence

$$\bigcap_{x \in S} I(x) \subseteqq \{y \mid y \leqq s\} = I(s).$$

Thus, $f(\wedge S) = \cap f[S]$ and f is infima preserving. From this, in view of the two equalities used in the proof of Lemma 1 it follows that f is also suprema preserving. As indicated by (11), obviously, $\vee f[E] \supseteqq \cup f[E]$.

Since f is a one-to-one suprema and infima preserving mapping, it is customary to refer to $(L, \subseteqq)$ as the *conditional-completion by lower cuts of* $(P, \leqq)$. In $(L, \subseteqq)$, every element x of P is identified with the lower segment $I(x)$ of P.

Remark 1. A partially ordered set is called *complete* if and only if every subset of it has an infimum (or equivalently, has a supremum). As usual, by adjoining at most two elements to a conditional-completion of a partially ordered set $(P, \leqq)$, a *completion* of $(P, \leqq)$ can be obtained. Thus, if $(P, \leqq)$ has neither a least nor a greatest element, then by adjoining $\emptyset$ and P respectively to $(L, \subseteqq)$, a completion of $(P, \leqq)$ is obtained.

3. Conditional-completion of Boolean rings

In this section we show that if instead of an arbitrary partially ordered set we start with a Boolean ring $(B, \leqq)$, then the conditional-completion $(L, \subseteqq)$ by lower cuts of $(B, \leqq)$ is again a Boolean ring.

We recall that a Boolean ring is a *distributive lattice which is closed under subtraction* (see (1), (2), (3)).

Let us recall also that a lattice $(M, \leqq)$ with least element 0 is called *sectionally complemented* [3, p. 28] *if and only if for every element x and u of M such that $x \leqq u$, there exists an element y of M such that*

$$x \vee y = u \quad and \quad x \wedge y = 0 \tag{14}$$

in which case y is called *a complement of x with respect to u.* It is easily seen that if $(M, \leqq)$ is a distributive lattice then an element x of M has at most one complement y with respect to an element u of M.

Although we define a Boolean ring as a distributive lattice which is closed under subtraction, as shown by the lemma below, we can also regard a Boolean ring as a sectionally complemented distributive lattice (and equivalently, [3, p. 48] or [1, p. 130] as a *relatively complemented distributive lattice with least element* 0).

Lemma 3. *A distributive lattice $(B, \leqq)$ is a Boolean ring if and only if $(B, \leqq)$ is sectionally complemented.*

Proof. Let $(B, \leqq)$ be a Boolean ring, and let x and u be elements of B such that $x \leqq u$. Then clearly, the difference $u - x$, as given by (3), is the complement of x with respect to u. Thus $(B, \leqq)$ is sectionally complemented.

Next, let $(B, \leqq)$ be a sectionally complemented distributive lattice with least element 0, and let x and y be elements of B. Let d be the complement of y with respect to $x \vee y$. Thus, $d \vee y = x \vee y$ and $d \wedge y = 0$, and therefore, in view of (3), we see that d is the difference $x - y$. Hence, $(B, \leqq)$ is closed under subtraction and the lemma is proved.

As mentioned earlier, we show in this paper that the conditional completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ is a (conditionally complete) Boolean ring. To this end, in view of the above, it is enough to prove that the conditional completion $(L, \subseteqq)$ of a sectionally complemented distributive lattice $(B, \leqq)$ is again a sectionally complemented distributive lattice. In what follows, we shall prove this by a sequence of lemmas and partial results. Many of the lemmas are established to prove the distributivity of $(L, \subseteqq)$. It seems that no shorter and direct proof of the distributivity of $(L, \subseteqq)$ is plausible.

Notation. *In what follows, we let* $(L, \subseteqq)$ *represent the conditional completion by lower cuts of a Boolean ring* $(B, \leqq)$.

Lemma 4. $(L, \subseteqq)$ *is a lattice with least element* $\{0\}$.

PROOF. Obviously $\{0\}$ is the least element of $(L, \subseteqq)$ because 0 is an element of every lower cut of B. Thus to prove the lemma, it remains to show that if X and Y are elements of L, then X and Y have a supremum and an infimum in L.

Let X and Y be elements of L. Then X and Y can be given in the form $X = \bigcap_{v \in M} I(v)$ and $Y = \bigcap_{v \in N} I(v)$, and let $m \in M$ and $n \in N$. Since B is a Boolean ring and $M \subseteqq B$ and $N \subseteqq B$, it is obvious that $(m \vee n) \in B$ and therefore by (12) we have $I(m \vee n) \in L$. But then $X \subseteqq I(m \vee n)$ and $Y \subseteqq I(m \vee n)$. Consequently X and Y are bounded above in $(L, \subseteqq)$ by $I(m \vee n)$, and thus have a supremum in $(L, \subseteqq)$ since $(L, \subseteqq)$ is conditionally complete.

On the other hand, from (10) it is clear that $\bigcap_{v \in (M \cup N)} I(v)$ is the infimum $X \wedge Y = X \cap Y$ of X and Y in $(L, \subseteqq)$.

Hence the lemma is proved.

Lemma 5. $(L, \subseteqq)$ *is sectionally complemented.*

PROOF. Let X and Y be elements of L such that $X = \bigcap_{v \in M} I(v)$ where M is the set of *all* upper bounds of X and $Y = \bigcap_{v \in N} I(v)$ where N is the set of *all* upper bounds of Y. Moreover let

$$X \subseteqq Y. \tag{15}$$

To prove the lemma, it is sufficient to show that the lower cut D of B given by

$$D = \cap \{I(v) \mid v = (n-x) \text{ for some } n \in N \text{ and some } x \in X\} \tag{16}$$

is a complement of X with respect to Y in $(L, \subseteqq)$. In the above, the difference $n-x$ exists since B is a Boolean ring and $N \subseteqq B$ and $X \subseteqq B$.

In view of (14), we must show that

$$X \cap D = \{0\} \tag{17}$$

and

$$X \vee D = Y. \tag{18}$$

Let $x \in (X \cap D)$. Then $x \leqq (n-x)$ for some $n \in N$, which implies $x=0$ since B is a Boolean ring. Thus (17) is established.

We note that $D \subseteqq Y$ because $v \in D$ implies $v \leqq (n-x)$ for every $n \in N$ and every $x \in X$. Hence $v \leqq n$ for every $n \in N$, and thus $v \in Y$. Consequently from (17) it follows that

$$D \subseteqq ((Y \setminus X) \cup \{0\}). \tag{19}$$

We claim that an element u of B is an upper bound of $X \cup D$ if and only if $u \in N$. IF $u \in N$ then u is an upper bound of Y and hence u is an upper bound of $X \cup D$ since $(X \cup D) \subseteqq Y$ as shown by (15) and (19).

Next, let u be an upper bound of $X \cup D$ and let us assume, on the contrary, that $u \notin N$. Then since u is an upper bound of X, we see that $u \in M$ and consequently by our assumpion, $u \in (M \setminus N)$. However, since $u \notin N$, there exists $y \in (Y \setminus X)$ such that

$$y \not\leqq u. \tag{20}$$

We show that $(y-u) \in D$. Since u is an upper bound of X we have $x \leqq u$ for every $x \in X$. Furthermore, since B is a Boolean ring, we have $(n-u) \leqq (n-x)$ for every $x \in X$ and every $n \in N$. Since $y \leqq n$ for every $n \in N$ and since B is a Boolean ring, it is clear that $(y-u) \leqq (n-u) \leqq (n-x)$ for every $x \in X$ and every $n \in N$. Hence $(y-u) \in D$. But then, since u is an upper bound of $X \cup D$ and hence is an upper bound of D, we have $u \geqq (y-u)$, which implies $y \leqq u$ since B is a Boolean ring. This contradicts (20). Hence our assumption is false, and $u \in N$.

Thus an element u of B is an upper bound of $(X \cup D)$ if and only if $u \in N$, and therefore $X \vee D = \bigcap_{v \in N} I(v) = Y$, and (18) is established. Consequently D is a complement of X with respect to Y, and the lemma is proved.

From Lemmas 4 and 5 we have

Corollary 1. *The conditional completion $(L, \subseteqq)$ by lower cuts of Boolean ring $(B, \leqq)$ is a sectionally complemented lattice.*

Notation. *Let X and Y be elements of $(L, \subseteqq)$ such that $X \subseteqq Y$. Then the complement D of X with respect to Y as given by* (16) *is denoted by*

$$Y-X. \tag{21}$$

We recall that a Boolean ring $(B, \leqq)$ is also a ring $(B, +, \cdot)$ in which every element is idempotent, and therefore $(B, +, \cdot)$ is commutative and of characteristic 2. In other words, for every element x and y of a Boolean ring $(B, +, \cdot)$, we have:

$$xy = yx \quad \textit{and} \quad x^2 = x \quad \textit{and} \quad x+x = 0. \tag{22}$$

Moreover, for every element x and y of a Boolean ring $(B, \leqq) \equiv (B, +, \cdot)$ we have:

$$x \leqq y \quad \textit{if and only if} \quad xy = x \tag{23}$$

and

$$y-x = y+yx, \quad \textit{and,} \quad y-x = y+x \quad \textit{if and only if} \quad x \leqq y. \tag{24}$$

It is worth observing that the passage from $(B, \leqq)$ to $(B, +, \cdot)$ is given by

$$x+y = (x \vee y)-(x \wedge y) \quad \text{and} \quad xy = x \wedge y.$$

Next we prove

Lemma 6. *For every element X and Y of $(L, \subseteqq)$ such that $X \subseteqq Y$ we have*

$$Y-(Y-X) = X. \tag{25}$$

PROOF. Let $X = \bigcap_{v \in M} I(v)$ where M is the set of all upper bounds of X, and let $Y = \bigcap_{v \in N} I(v)$ where N is the set of all upper bounds of Y. From (16) and (21) it follows that

$$(26) \quad Y-(Y-X) = \cap \{I(v) \mid v = (n-z) \textit{ for some } n \in N \textit{ and for some } z \in (Y-X)\}.$$

First we show that if $w \in X$ then $w \in (Y-(Y-X))$. If $w \in X$ then w is a lower bound of N and therefore $w \leqq n$ for every $n \in N$. On the other hand, since B is a Boolean ring, then for every $z \in (Y-X)$, in view of (24), we have $w(n-z) = w - wz = w$ since, by (17) and (21) we have $X \cap (Y-X) = \{0\}$ and therefore $wz = 0$. Thus by, (23) we see that $w \leqq (n-z)$ for every $n \in N$ and every $z \in (Y-X)$. Consequently, $w \in (Y-(Y-X))$, by (26).

Next, let w be an element of $Y-(Y-X)$. We show that $w \in X$. Let us assume, on the contrary, that $w \notin X$. Clearly $w \in Y$ and consequently, by our assumption, $w \in (Y \setminus X)$. However, since $w \notin X$, there exists $u \in (M \setminus N)$ such that

$$(27) \qquad w \not\leqq u.$$

Since u is an upper bound of X, we see that $x \leqq u$ for every $x \in X$ and therefore since B is a Boolean ring, $(n-u) \leqq (n-x)$ for every $x \in X$ and every $n \in N$. Again, since $w \leqq n$, we have $(w-u) \leqq (n-u) \leqq (n-x)$ for every $x \in X$ and every $n \in N$. Consequently, $(w-u) \in (Y-X)$. However, since $w \in (Y-(Y-X))$ we see that $w \leqq (n-(w-n))$ for some $n \in N$. But this, in view of (23), (24) and (22) implies $wn + w + wu = w$ and since $w \leqq u$, it follows that $w = wu$. Hence, by (23) we see that $w \leqq u$ which contradicts (27). Thus, our assumption is false and indeed $w \in X$, and the lemma is proved.

Lemma 7. *For every element X and Y of $(L, \subseteq)$ such that $X \subseteq Y$ we have*

$$(28) \qquad t \in (Y-X) \quad \textit{if and only if} \quad t \in Y \quad \textit{and} \quad tx = 0 \quad \textit{for every} \quad x \in X$$

and

$$(29) \qquad t \in X \quad \textit{if and only if} \quad t \in Y \quad \textit{and} \quad tc = 0 \quad \textit{for every} \quad c \in (Y-X).$$

PROOF. Let $t \in (Y-X)$. By (17) and (21) we have $(Y-X) \cap X = \{0\}$, and hence $tx = 0$ for every $x \in X$ since $tx \in ((Y-X) \cap X)$. This is because $Y-X$, as well as X, is a lower cut of $(L, \subseteq)$.

Next let $t \in Y$ and $tx = 0$ for every $x \in X$. Let us assume, on the contrary, that $t \notin (Y-X)$. Thus by (16) we see that $t \not\leqq (n-x)$ for some $n \in N$ and some $x \in X$ where $Y = \bigcap_{v \in N} I(v)$ and where N is the set of all upper bounds of Y. Therefore by (23) and (24) we have $t(n+x) \neq t$ and hence $(tn + tx) \neq t$. However, since $t \in Y$ implies $t \leqq n$, we have $tn = t$ which implies $tx \neq 0$, contradicting the hypothesis that $tx = 0$ for every $x \in X$. Hence our assumption is false and (28) is established.

In order to prove (29), let us observe that by (28) we have $t \in (Y-(Y-X))$ if and only if $t \in Y$ and $tc = 0$ for every $c \in (Y-X)$, which in view of (25) implies (29).

Lemma 8. *For every element X, Y and Z of $(L, \subseteq)$ such that $X \subseteq Z$ and $Y \subseteq Z$, we have*

$$(30) \qquad X \subseteq Y \quad \textit{if and only if} \quad (Z-X) \supseteq (Z-Y).$$

PROOF. Let $X \subseteqq Y$ and let $t \in (Z-Y)$. Then by (28), we have $ty=0$ for every $y \in Y$, and since $X \subseteqq Y$ we have $tx=0$ for every $x \in X$. Thus again by (28) we have $t \in (Z-X)$.

Next let $(Z-X) \supseteqq (Z-Y)$, and let $t \in X$. Then by (29) we see that $tc=0$ for every $c \in (Z-X)$. Since $(Z-X) \supseteqq (Z-Y)$ we have $td=0$ for every $d \in (Z-Y)$. But then again by (29) we have $t \in Y$, and hence $X \subseteqq Y$.

Lemma 9. *For every element X, Y and Z of $(L, \subseteqq)$ such that $X \subseteqq Z$ and $Y \subseteqq Z$ we have*

$$X \subseteqq Y \quad \text{if and only if} \quad X \cap (Z-Y) = \{0\}. \tag{31}$$

PROOF. Let $X \subseteqq Y$. By (30) we have $(Z-X) \supseteqq (Z-Y)$, and by (17) and (21) we see that $\{0\} = (X \cap (Z-X)) \supseteqq (X \cap (Z-Y))$, and therefore $X \cap (Z-Y) = \{0\}$.

Next let $X \cap (Z-Y) = \{0\}$, and let $x \in X$. But then $xc=0$ for every $c \in (Z-Y)$ since $xc \in (X \cap (Z-Y))$. This is because X as well as $Z-Y$ is a lower cut of $(L, \subseteqq)$. Therefore by (29) we see that $x \in Y$, and hence (31) is established.

Lemma 10. *For every element X, Y and Z of $(L, \subseteqq)$ such that $X \subseteqq Z$ and $Y \subseteqq Z$ we have*

$$Z-(X \wedge Y) = (Z-X) \vee (Z-Y) \tag{32}$$

and

$$Z-(X \vee Y) = (Z-X) \wedge (Z-Y). \tag{33}$$

PROOF. Since $X \subseteqq (X \vee Y)$ and $Y \subseteqq (X \vee Y)$ it follows from (30) that $(Z-(X \vee Y)) \subseteqq (Z-X)$ and $(Z-(X \vee Y)) \subseteqq (Z-Y)$ and consequently $(Z-(X \vee Y)) \subseteqq (Z-X) \cap (Z-Y) = (Z-X) \wedge (Z-Y)$.

Thus to prove (32), it remains to show that

$$(Z-(X \vee Y)) \supseteqq (Z-X) \wedge (Z-Y). \tag{34}$$

But then in view of (31), to establish (34), it is sufficient to show that

$$((Z-X) \wedge (Z-Y)) \cap (Z-(Z-(X \vee Y))) = \{0\}$$

which in view of (10) and (25) reduces to proving that

$$(Z-X) \cap (Z-Y) \cap (X \vee Y) = \{0\}. \tag{35}$$

Let us assume the contrary, and let

$$(Z-X) \cap (Z-Y) \cap (X \vee Y) = C \neq \{0\} \tag{36}$$

for some lower cut C of $(L, \subseteqq)$. We claim that

$$X \subseteqq ((X \vee Y)-C). \tag{37}$$

Let $t \in X$. Then, since $X \subseteqq Z$, by (29) we see that $tx=0$ for every $x \in (Z-X)$. Hence, $tc=0$ for every $c \in C$, since $C \subseteqq (Z-X)$, by (36). Again, by (36) we have $C \subseteqq (X \vee Y)$ and therefore, from (28) it follows that $t \in ((X \vee Y)-C)$, since $t \in X$ and $X \subseteqq (X \vee Y)$ and $tc=0$ for every $c \in C$. Thus, indeed $X \subseteqq ((X \vee Y)-C)$, and (37) is established.

Similarly, we can prove that

$$Y \subseteqq ((X \vee Y) - C). \tag{38}$$

From (37) and (38) it follows that

$$(X \vee Y) \subseteqq ((X \vee Y) - C) \subseteqq (X \vee Y)$$

which implies $C = \{0\}$ since $C \subseteqq (X \vee Y)$. This contradicts (36), and therefore (35), and hence also (32), is established.

To prove (33), we substitute $Z - X$ for X and $Z - Y$ for Y in (32), and in view of (25) we obtain $X \vee Y = Z - ((Z - X) \wedge (Z - Y))$ which, again in view of (25), yields (33), as desired.

A dual of Lemma 9 is given by

Corollary 2. *For every element X, Y and Z of $(L, \subseteqq)$ such that $X \subseteqq Z$ and $Y \subseteqq Z$ we have*

$$X \subseteqq Y \quad \textit{if and only if} \quad ((Z - X) \vee Y) = Z. \tag{39}$$

Proof. In view of (31) we have

$$X \subseteqq Y \quad \text{if and only if} \quad Z - (X \wedge (Z - Y)) = Z - \{0\} = Z \tag{40}$$

which by (33) and (25) implies $X \subseteqq Y$ if and only if $(Z - X) \vee (Z - (Z - Y)) = (Z - X) \vee$ $\vee Y = Z$, as desired.

Based on the above lemmas, we prove the following theorem.

Theorem 2. *For every element X, Y and Z of $(L, \subseteqq)$ we have*

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z) \tag{41}$$

and

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z). \tag{42}$$

Proof. Since $X \wedge (Y \vee Z) \supseteqq (X \wedge Y)$ and $X \wedge (Y \vee Z) \supseteqq (X \wedge Z)$ it is clear that $X \wedge (Y \vee Z) \supseteqq ((X \wedge Y) \vee (X \wedge Z))$. Hence to establish (41) it remains to show that

$$X \wedge (Y \vee Z) \subseteqq ((X \wedge Y) \vee (X \wedge Z)). \tag{43}$$

Let $H = X \vee Y \vee Z$. Then

$$\begin{aligned} & X \wedge (Y \vee Z) \wedge (H - ((X \wedge Y) \vee (X \wedge Z))) \wedge Y = \\ & = X \wedge (Y \vee Z) \wedge (H - (X \wedge Y)) \wedge (H - ((X \wedge Z))) \wedge Y = \qquad \text{by (33)} \\ & = ((X \wedge Y) \wedge (H - (X \wedge Y))) \wedge (Y \vee Z) \wedge (H - (X \wedge Z)) = \{0\} \ \text{by (17) and (21).} \end{aligned} \tag{44}$$

Therefore, by (25) we have:

$$X \wedge (Y \vee Z) \wedge (H - ((X \wedge Y) \vee (X \wedge Z))) \wedge (H - (H - Y)) = \{0\}$$

which by (31 implies:

$$X \wedge (Y \vee Z) \wedge (H - ((X \wedge Y) \vee (X \wedge Z))) \subseteqq (H - Y). \tag{45}$$

Similarly, by replacing the last Y appearing in (44) by Z, we obtain

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z)))\wedge Z = \{0\}$$

which, as in the case of (45), implies

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq (H-Z). \tag{46}$$

But then, from (45) and (46) it follows that

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq ((H-Y)\wedge(H-Z))$$

which by (33) implies

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq (H-(Y\vee Z)). \tag{47}$$

On the other hand, it is clear that

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq (Y\vee Z)$$

which, in view of (47) and (17) implies

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq ((H-(Y\vee Z))\wedge(Y\vee Z)) = \{0\}$$

and consequently,

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) = \{0\}$$

which by (31) implies (43). But then (41) follows, as mentioned above.

To prove (42), we replace X, Y and Z respectively by $H-X$, $H-Y$ and $H-Z$ in (41). As a result, we obtain

$$(H-X)\wedge((H-Y)\vee(H-Z)) = ((H-X)\wedge(H-Y))\vee((H-X)\wedge(H-Z))$$

which by (32) and (33) implies

$$(H-X)\wedge(H-(Y\wedge Z)) = (H-(X\vee Y))\vee(H-(X\vee Z))$$

which, in turn, by (25), (32) and (33) implies

$$X\vee(Y\wedge Z) = H-((H-X)\wedge(H-(Y\wedge Z))) = H-((H-(X\vee Y))\vee(H-(X\vee Z))) =$$
$$=(X\vee Y)\wedge(X\vee Z)$$

as desired. Thus, (42) is established.

The above Theorem establishes the distributivity of $(L, \subseteqq)$. Observing that in a distributive lattice (sectional) complementation is unique, based of Lemmas 3, 4, 5 and Theorems 1 and 2, we have:

Theorem 3. *The conditional completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ is a sectionally complemented distributive lattice and hence a conditionally complete Boolean ring. Moreover, the mapping f from B into L defined by*

$$f(x) = I(x) = \{y \mid y\in B \text{ and } y \leqq x\}$$

is a ring isomorphism from $(B, \leqq)$ into $(L, \subseteqq)$ such that f preserves suprema (and hence, also infima).

Remark 2. Let us observe that if the Boolean ring $(B, \leqq)$ in Theorem 3 is a Boolean algebra (i.e., B has a multiplicative unit 1), then the conditional completion $(L, \subseteqq$ by) lower cuts of $(B, \leqq)$ is in fact a complete Boolean algebra (see Remark 1 on page 255). Thus, to obtain a completion of a Boolean ring $(B, \leqq)$ which is not a Boolean algebra, it is sufficient to consider the conditional completion by lower cuts of the Boolean algebra $(A, \leqq)$ which is obtained by adjoining to B a new symbol 1 (as a multiplicative unit) along with $x+1$ for every $x \in B$ and by defining addition and multiplication in A in an obvious way.

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IOWA STATE UNIVERSITY
AMES, IOWA

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