

A new algebra of distributions; initial-value problems involving Schwartz distributions. I

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The theme of this paper is a space $\mathfrak{B}$ of distributions; this space $\mathfrak{B}$ is closed under convolution (the definition of this particular convolution requires no restriction on the supports — nor does it require growth conditions). The space $\mathfrak{B}$ contains all the functions which are locally integrable on $(-\infty, \infty)$; the space $\mathfrak{B}$ also contains D'_+ and each derivative of every distribution whose support is a locally finite subset of $(-\infty, \infty)$. If T is a distribution whose derivative ∂T belongs to $\mathfrak{B}$, then T also belongs to $\mathfrak{B}$, and T has a well-defined *initial-value* $T(0-)$. Thus, it is possible to consider initial-value problems involving arbitrary distributions whenever the input belongs to the space $\mathfrak{B}$. Also defined in this paper is a one-to-one transformation of $\mathfrak{B}$ into a commutative algebra of operators (this transformation is somewhat analogous to the Fourier transformation); this gives an operational calculus which yields an existence and uniqueness theorem for differential equations subject to initial conditions of the form

$$u(0-) = c_0, \quad \partial u(0-) = c_1, \quad \partial^2 u(0-) = c_2, \dots,$$

where $c_0, c_1, c_2, \dots$ are arbitrary constants and where u is an arbitrary distribution satisfying a differential equation whose right-hand side belongs to the space $\mathfrak{B}$. The operational calculus is applied to a differential equation which cannot be solved explicitly by means of the Fourier transformation.

The new algebra is denoted $\mathfrak{B}$: it is a commutative algebra (of distributions) under convolution multiplication; the space $\mathfrak{B}$ contains D'_+ and all locally integrable functions. The space $\mathfrak{B}$ is closed under convolution and contains each derivative of every distribution whose support is locally finite. If F is a distribution whose derivative ∂F belongs to $\mathfrak{B}$, then F also belongs to $\mathfrak{B}$ and F equals a continuous function f in some interval $(a, 0)$ (with $a < 0$): in consequence, F has a well-defined initial value $F(0-) = f(0-)$.

Given arbitrary constants $(a_0, a_1, \dots, a_m)$ and an arbitrary element S of the algebra $\mathfrak{B}$, we shall describe a calculus to obtain explicit solutions of differential equations of the form

$$(1) \quad a_m \partial^m u + \dots + a_1 \partial u + a_0 u = S$$

subject to initial conditions such as

$$(2) \quad u(0-) = c_0, \quad \partial u(0-) = c_1, \dots, \partial^{m-1} u(0-) = c_{m-1},$$

where $c_0, c_1, \dots, c_{m-1}$ are arbitrary constants. As we shall see, such initial-value problems can be solved by means of an operational calculus which is a useful substitute for the two-sided distributional Laplace transformation: it requires *no growth restrictions* and *no restrictions on the supports*. We shall prove that the equation (1) implies that both u and $\partial^k u$ belong to $\mathfrak{B}$ for each integer $k \leq m$; moreover, there exists a unique distribution u satisfying the problem (1)–(2): this distribution belongs to the algebra $\mathfrak{B}$.

The algebra $\mathfrak{B}$ contains as a subalgebra the space D'_+ (of all the distributions which vanish on the interval $(-\infty, 0)$). In fact, $\mathfrak{B} = D'_+ + \mathfrak{B}_-$, where $\mathfrak{B}_-$ is the space of all the elements of D'_- which are regular in some neighborhood of the origin (here D'_- denotes the space of all the distributions which vanish on $(0, \infty)$). The space $\mathfrak{B}$ contains the space L^{loc} of all the complex-valued functions which are locally integrable on $(-\infty, \infty)$:

$$L^{\text{loc}} \subset \mathfrak{B} \subset D'.$$

The space $\mathfrak{B}$ also contains all the “opérateurs de Heaviside” [2]. To each F in $\mathfrak{B}$ there corresponds a unique pair (F_+, F_-) in the cartesian product $D'_+ \times \mathfrak{B}_-$ such that $F = F_+ + F_-$.

The algebra $\mathfrak{B}$ is the result of providing the space $\mathfrak{B}$ with the multiplication

$$\mathfrak{B} \times \mathfrak{B} \ni (F, G) \rightarrow F \otimes G \in \mathfrak{B}$$

defined by

$$F \otimes G = F_+ * G_+ - F_- * G_- ,$$

where $*$ is convolution in the usual sense [4, p. 348]. Thus, $\otimes$ is the multiplication of the algebra $\mathfrak{B}$; it is commutative and commutes with the distributional differentiator ∂ ; note the absence of any restriction on the supports (there are no growth conditions either). A particular solution of the differential equation (1) is given by the Duhamel-type formula

$$u = G \otimes S,$$

where G is the regular distribution corresponding to the Green's function of the equation (1); see 5.15.

If $[f]^0$ (respectively, $[g]^0$) is the regular distribution corresponding to the locally integrable function $f(\cdot)$ (respectively, $g(\cdot)$), then $[f]^0 \otimes [g]^0$ is the regular distribution $[f \wedge g]^0$ corresponding to the function $f \wedge g(\cdot)$ defined by

$$f \wedge g(t) = - \int_t^0 f(t-\tau)g(\tau) d\tau \quad (\text{for } -\infty < t < \infty).$$

Our *operational calculus* (see 3.8) is an algebraic isomorphism of $\mathfrak{B}$ into the operator-algebra $\mathcal{A}_R$ (see [6]); it applies to problems such as (1)–(2); for example, it yields the solution

$$(3) \quad u = c_0 \cos t + (c_1 + 1) \sin t + \left[\frac{t}{2\pi} \right] \sin t$$

of the initial-value problem

$$(4) \quad \partial^2 u + u = \sum_{k=-\infty}^{\infty} \delta_{2k\pi} \quad \text{subject to} \quad u(0-) = c_0$$

and $\partial u(0-) = c_1$ (with c_0 and c_1 arbitrary complex numbers): as usual, $[t/2\pi]$ is the greatest integer $< t/2\pi$, and $\delta_{2k\pi}$ is the Dirac distribution concentrated at the point $2k\pi$. The first equation in (4) is a counter-example in [1, p. 128] and cannot be solved by the method of fundamental (or "elementary") solutions; its solution (3) cannot be obtained by using the Fourier transformation, the finite Fourier transformation, nor the two-sided distributional Laplace transformation.

Organization of this paper. The first sections are devoted to the operational calculus; § 5 deals with initial-value problems and contains the main results; the above example is discussed in § 6.

Concluding remarks. The theorems in § 5 resemble (and were inspired by) the ones in César de Freitas' article [2] (which deals with a space $\mathfrak{M}$ containing L^{loc} and properly contained in $\mathfrak{B}$; each element of $\mathfrak{M}$ is a linear combination of a function with a sum of distributions of finite order whose supports are locally finite). Harris Shultz gave me the idea that $F \otimes G$ belongs to $\mathfrak{B}$ whenever F and G belong to $\mathfrak{B}$; he also gave me a more elegant characterization of the space $\mathfrak{B}$ (which I use as a definition).

§ 1. Preliminaries

Let D be the space $D(\mathbf{R})$ of Schwartz test-functions on $\mathbf{R} = (-\infty, \infty)$; as usual, D' is the dual of D (see [4, p. 313]); a *distribution* is an element of D' .

If J is a subset of $\mathbf{R}$, the relation $\varphi < J$ will mean that $\varphi(\cdot) \in D$ and the support of $\varphi(\cdot)$ is a compact subset of J . We always have $\varphi < \mathbf{R}$.

1.1. *The symbol $\bigcirc$.* If F is a distribution, there exists a largest open set $\bigcirc(F)$ such that $F(\varphi) = 0$ whenever $\varphi < \bigcirc(F)$ (see [4, p. 318]).

1.2. If (α, β) is an open interval, then $\varphi < (\alpha, \beta)$ if (and only if) $\varphi(\cdot)$ vanishes outside some closed sub-interval of (α, β) .

1.3. Again, let F be a distribution. If J_1 and J_2 are open subsets of $\mathbf{R}$ such that $\bigcirc(F) \supset J_k$ for $k=1$ and $k=2$, then

$$\bigcirc(F) \supset J_1 \cup J_2 \quad (\text{see [8, pp. 27—28]}).$$

1.4. *Equality of distributions in a set.* If $J \subset \mathbf{R}$, a distribution F is said to *equal* F_1 in J if (and only if) $F(\varphi) = F_1(\varphi)$ whenever $\varphi < J$.

Thus, $\bigcirc(F)$ is the largest open set J such that F equals $\mathbf{0}$ in J . Note that $\mathbf{0}$ is the zero distribution.

1.5. *Note.* Let J be an open subset of $\mathbf{R}$; clearly,

$$\bigcirc(F) \supset J \quad \text{if (and only if) } F \text{ equals } \mathbf{0} \text{ in } J.$$

1.6. **Lemma.** Let c_1 and c_2 be complex numbers. If F_1 and F_2 are distributions, then

$$\bigcirc(c_1 F_1 + c_2 F_2) \supset \bigcirc(F_1) \cap \bigcirc(F_2):$$

see [4, p. 318, Proposition 2].

1.7. *Two spaces of regular distributions.* Let $\mathcal{F}$ be the space L^{loc} of all the complex-valued functions which are locally integrable on $\mathbf{R}$. There is a linear injection $f() \rightarrow [f]^0$ of $\mathcal{F}$ into the space D' of distributions, the distribution $[f]^0$ being defined by

$$[f]^0(\varphi) = \int_{-\infty}^{\infty} f(u)\varphi(u) du \quad (\text{for } \varphi() \text{ in } D):$$

see p. 48 in [9], where the distribution $[f]^0$ is denoted T_f . If $\mathcal{G} \subset \mathcal{F}$ we set

$$[\mathcal{G}]^0 = \{[f]^0: f() \in \mathcal{G}\}.$$

The elements of the space $[\mathcal{F}]^0$ are usually called *regular distributions*. Let $\mathcal{F}_-$ be the space of all the functions $f()$ in $\mathcal{F}$ such that $f()$ equals zero almost-everywhere on $(0, \infty)$ (this will henceforth be written: $f()=0$ on $(0, \infty)$):

$$(1.8) \quad \mathcal{F}_- = \{f() \in \mathcal{F}: f() = 0 \text{ on } (0, \infty)\}.$$

Thus, $B \in [\mathcal{F}_-]^0$ if B is a regular distribution $[f]^0$ such that $f()=0$ on $(0, \infty)$:

$$(1.9) \quad [\mathcal{F}_-]^0 = \{[f]^0: f() \in \mathcal{F} \text{ and } f() = 0 \text{ on } (0, \infty)\}.$$

1.10. Let Ω be an open subset of $\mathbf{R}$ and let $f_1()$ and $f_2()$ be two elements of $\mathcal{F}$. If $f_1()=f_2()$ on Ω , then the distribution $[f_1]^0$ equals $[f_2]^0$ in Ω . See [9, p. 48].

1.11. Conversely, if $[f_1]^0$ equals $[f_2]^0$ in Ω , then $f_1()=f_2()$ on Ω (that is, the functions are equal almost-everywhere on Ω). See [9, p. 48].

1.12. Definition. Let $\mathfrak{B}_-$ be the family of all the distributions which equal $\mathbf{0}$ in the interval $(0, \infty)$ and which equal some regular distribution in some interval $(a, -a)$ with $a < 0$.

1.13. Definitions. Let $\mathfrak{B}_+$ be the family of all the distributions which equal $\mathbf{0}$ in the interval $(-\infty, 0)$. We set

$$(1.14) \quad \mathfrak{B} = \mathfrak{B}_- + \mathfrak{B}_+;$$

thus, the family $\mathfrak{B}$ consists of all the sums $F+R$ such that $F \in \mathfrak{B}_-$ and $R \in \mathfrak{B}_+$.

1.15. Definition. A "left-distribution" is a distribution which equals zero in some interval (a, ∞) containing the point 0. We denote by $(\mathcal{L})$ the family of left-distributions.

1.16. Therefore, $L \in (\mathcal{L})$ if (and only if) L is a distribution which equals $\mathbf{0}$ in some interval (a, ∞) with $a < 0$. In other words, $L \in (\mathcal{L})$ if (and only if) $L \in D'$ and there exists a number $a < 0$ such that $\circ(L) \supset (a, \infty)$.

1.17. Reorientation. The main object of this section is to prove that 1.14 is a direct sum. We shall also need the following lemma, which characterizes $\mathfrak{B}_-$ as the space of all distributions of the form $L+[f]^0$, where $L \in (\mathcal{L})$ and $f() \in \mathcal{F}_-$.

1.18. Lemma. $\mathfrak{B}_- = (\mathcal{L}) + [\mathcal{F}_-]^0$.

PROOF. If A belongs to $(\mathcal{L}) + [\mathcal{F}_-]^0$, then the equation

$$(1) \quad A = L + [f]^0$$

holds for some $L \in (\mathcal{L})$ and for some $f(\cdot) \in \mathcal{F}_-$. Since $f(\cdot) = 0$ on $(0, \infty)$ (by 1.8), it follows from 1.10 and 1.5 that

$$(2) \quad \circ([f]^0) \supset (0, \infty).$$

Since $L \in (\mathcal{L})$ we can apply 1.16 to infer the existence of a number $a < 0$ such that

$$(3) \quad \circ(L) \supset (a, \infty).$$

In view of (1), (2) and (3), we may apply 1.6 to obtain

$$(4) \quad \circ(A) \supset (0, \infty):$$

this is obtained by using the fact that $a < 0$, which implies the equation $(0, \infty) \cap \cap(a, \infty) = (0, \infty)$.

Thus, A equals $\mathbf{0}$ in $(0, \infty)$: we still have to prove that A equals $[f]^0$ in $(a, -a)$. To that effect, observe that L equals $\mathbf{0}$ in (a, ∞) (by (3) and 1.5); from (1) we therefore infer that A equals $[f]^0$ in (a, ∞) . Consequently, A belongs to $\mathfrak{B}_-$ (since A also equals $\mathbf{0}$ in $(0, \infty)$ (see (4) and 1.12)).

To prove the converse, suppose that $A \in \mathfrak{B}_-$. In view of 1.12, this implies (4), the existence of a number $a < 0$ and a function $g(\cdot)$ in $\mathcal{F}$ such that

$$(5) \quad A(\varphi) = [g]^0(\varphi) \quad (\text{for } \varphi < (a, -a)).$$

Since $\circ(A) \supset (0, -a)$ (by (4) and $a < 0$), we have

$$(6) \quad A(\varphi) = 0 \quad (\text{for } \varphi < (0, -a)).$$

Combining (5) and (6), we see that

$$[g]^0(\varphi) = 0 \quad (\text{for } \varphi < (0, -a));$$

that is, $[g]^0$ equals $\mathbf{0}$ in $(0, -a)$; therefore, 1.11 gives

$$(7) \quad g(\cdot) = 0 \quad \text{on } (0, -a).$$

Let $f(\cdot)$ be the function defined by

$$(8) \quad f(t) = \begin{cases} g(t) & \text{for } a < t < -a \\ 0 & \text{otherwise.} \end{cases}$$

From (7) and (8) it follows that $f(\cdot) = 0$ on $(0, -a)$; since $f(\cdot) = 0$ on $(-a, \infty)$ (by (8)), we see that $f(\cdot) = 0$ on $(0, \infty)$: therefore,

$$(9) \quad \circ([f]^0) \supset (0, \infty),$$

which (by 1.11) implies that $f(\cdot) = 0$ on $(0, \infty)$; consequently, $f(\cdot)$ belongs to $\mathcal{F}_-$ and

$$(10) \quad [f]^0 \in [\mathcal{F}_-]^0.$$

Next, we set

$$(11) \quad L = A - [f]^0:$$

since $A = L + [f]^0$, our conclusion (namely, that A belongs to the space $(\mathcal{L}) + [\mathcal{F}_-]^0$) will be obtained by proving that $L \in (\mathcal{L})$. To that effect, observe that the equations

$$(12) \quad L(\varphi) = A(\varphi) - [f]^0(\varphi) = [g]^0(\varphi) - [g]^0(\varphi)$$

hold for $\varphi < (a, -a)$ and come from (11) and (5). Since $g(\cdot) = f(\cdot)$ on $(a, -a)$ (by (8)), the distribution $[g]^0$ equals $[f]^0$ in $(a, -a)$ (by 1.10), so that $[g]^0(\varphi) = [f]^0(\varphi)$ for $\varphi < (a, -a)$: from (12) it now follows that

$$L(\varphi) = 0 \quad (\text{for } \varphi < (a, -a)),$$

that is,

$$(13) \quad \bigcirc(L) \supset (a, -a).$$

On the other hand, (11), (4), (9) and 1.6 imply that

$$(14) \quad \bigcirc(L) \supset \bigcirc(A) \cap \bigcirc([f]^0) \supset (0, \infty).$$

From (13)—(14) and 1.3 it therefore follows that

$$\bigcirc(L) \supset (a, -a) \cup (0, \infty) = (a, \infty),$$

which proves that $\bigcirc(L) \supset (a, \infty)$: therefore, $L \in (\mathcal{L})$ (see 1.15). Since $A = L + [f]^0$ and (10), the distribution A belongs to $(\mathcal{L}) + [\mathcal{F}_-]^0$.

1.19. Lemma. Suppose that $B \in \mathfrak{B}_-$. If $B \in \mathfrak{B}_+$ then $B = \mathbf{0}$.

PROOF. If $B \in \mathfrak{B}_+$ it follows from 1.13 that B equals $\mathbf{0}$ in the interval $(-\infty, 0)$: this means that

$$(1) \quad B(\varphi) = 0 \quad (\text{for } \varphi < (-\infty, 0)).$$

In view of 1.18, the hypothesis $B \in \mathfrak{B}_-$ implies that the equation

$$(2) \quad B = L + [f]^0$$

holds for some L in $(\mathcal{L})$ and for some $f(\cdot)$ in $\mathcal{F}_-$. From (1)—(2) it follows that

$$(3) \quad [f]^0(\varphi) = -L(\varphi) \quad (\text{for } \varphi < (-\infty, 0)).$$

On the other hand, it follows from $L \in (\mathcal{L})$ and 1.16 the existence of a number $a < 0$ such that

$$(4) \quad L(\varphi) = 0 \quad (\text{for } \varphi < (a, \infty)).$$

Combining (3) and (4):

$$[f]^0(\varphi) = 0 \quad (\text{for } \varphi < (a, 0)),$$

so that $[f]^0$ equals $\mathbf{0}$ in $(a, 0)$, whence $f(\cdot) = 0$ on $(a, 0)$ (by 1.11). But our hypothesis $f(\cdot) \in \mathcal{F}_-$ implies that $f(\cdot) = 0$ on $(0, \infty)$; therefore, $f(\cdot) = 0$ on (a, ∞) , so that

$$(5) \quad [f]^0(\varphi) = 0 \quad (\text{for } \varphi < (a, \infty)).$$

From (5) and (4) we see that $L+[f]^0$ equals $\mathbf{0}$ in (a, ∞) ; from (2) it therefore follows that

$$\circ(B) \supset (a, \infty);$$

but $\circ(B) \supset (-\infty, 0)$ (from (1)), so that 1.3 now gives

$$(6) \quad \circ(B) \supset (-\infty, 0) \cup (a, \infty) = \mathbf{R}.$$

From (6), 1.5, and 1.1 we see that $B(\varphi)=0$ whenever $\varphi < \mathbf{R}$. If $\varphi(\cdot) \in D$ then $\varphi < \mathbf{R}$, so that the equation $B(\varphi)=\mathbf{0}(\varphi)$ holds for every $\varphi(\cdot)$ in D . We have proved that $B=\mathbf{0}$.

1.20. Theorem. *The four spaces $\mathfrak{B}_+$, $(\mathcal{L})$, $\mathfrak{B}_-$, and $\mathfrak{B}$ are linear spaces. To any F in $\mathfrak{B}$ there corresponds a unique pair (F_-, F_+) of distributions such that $F_- \in \mathfrak{B}_-$, $F = F_- + F_+$, and $F_+ \in \mathfrak{B}_+$. Moreover,*

$$(1.21) \quad F \in \mathfrak{B}_+ \Leftrightarrow F_- = \mathbf{0} \Leftrightarrow F = F_+,$$

and

$$(1.22) \quad F \in \mathfrak{B}_- \Leftrightarrow F_+ = \mathbf{0} \Leftrightarrow F = F_-.$$

PROOF. The linearity of $\mathfrak{B}_+$ and $(\mathcal{L})$ follow directly from 1.13, 1.15, and 1.6; it is now easy to verify that the space $(\mathcal{L}) + [\mathcal{F}_-]^0$ is linear; therefore, $\mathfrak{B}_-$ is linear (by 1.18); this in turn implies the linearity of the space $\mathfrak{B}$ (defined by $\mathfrak{B} = \mathfrak{B}_- + \mathfrak{B}_+$). Since $\mathfrak{B}$ consists of distributions of the form $B+R$ with $B \in \mathfrak{B}_-$ and $R \in \mathfrak{B}_+$, the uniqueness of the pair (B, R) will be established by proving that the assumption

$$(1) \quad F_- + F_+ = B + R$$

implies $F_- = B$ and $F_+ = R$ (when F_- and B belong to $\mathfrak{B}_-$, and when F_+ and R belong to $\mathfrak{B}_+$). From (1) it follows that $F_- - B$ belongs to both $\mathfrak{B}_-$ and $\mathfrak{B}_+$; from 1.19 it therefore follows that $F_- - B = \mathbf{0}$, which implies $F_- = B$ and $F_+ = R$ (by (1)).

To prove (1.21), note that $F \in \mathfrak{B}_+$ implies $F_- + F_+ \in \mathfrak{B}_+$, so that $F_- \in \mathfrak{B}_+$; since $F_- \in \mathfrak{B}_-$ we have $F_- = \mathbf{0}$ (by 1.19): the rest is obvious. The proof of 1.22 is entirely similar.

1.23. Notation. Let $1_-(\cdot)$ and $1_+(\cdot)$ be the functions defined by

$$(1.24) \quad 1_-(t) = \begin{cases} 1 & \text{for } t < 0 \\ 0 & \text{for } t \geq 0 \end{cases}$$

and

$$(1.25) \quad 1_+(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0. \end{cases}$$

If $f(\cdot) \in \mathcal{F}$ we set

$$(1.26) \quad f_-(\cdot) = 1_-(\cdot)f(\cdot) \quad \text{and} \quad f_+(\cdot) = 1_+(\cdot)f(\cdot).$$

1.27. Lemma. *If $f(\cdot) \in \mathcal{F}$ then $[f]^0 \in \mathfrak{B}$,*

$$(1.28) \quad [f_-]^0 = [f]_-^0, \quad \text{and} \quad [f_+]^0 = [f]_+^0.$$

PROOF. From 1.24—1.26 and 1.10 we see that $[f_-]^0 \in \mathfrak{B}$ and $[f_+]^0 \in \mathfrak{B}_+$; from 1.24—1.26 it also follows that $[f]^0 = [f_-]^0 + [f_+]^0$; the conclusions are now immediate from 1.20.

§ 2. The operation $\otimes$

In this section our aim is to define a multiplication $\otimes$ on the linear space $\mathfrak{B}$; its main properties will be established. First, let us set down some notation.

If R is a distribution, its distributional derivative ∂R is defined in the usual way [4, p. 323]. We denote by $1(\cdot)$ the constant function whose value is the unit 1 (the function $1(\cdot)$ is defined by $1(t) = 1$ for $t \in \mathbf{R}$). Note that $\partial[1]^0 = 0$,

$$(2.1) \quad \partial[1]_-^0 = -\delta, \quad \text{and} \quad \partial[1]_+^0 = \delta,$$

where $[1]_-^0 = [1_-]^0$ (see 1.28) and δ is the Dirac distribution. As usual, if $f(\cdot) \in \mathcal{F}$ then $\check{f}(\cdot)$ is the function defined by $\check{f}(x) = f(-x)$. If T is a distribution, then $\check{T}$ is the distribution (also denoted $T^\vee$) defined by

$$(2.2) \quad \check{T}(\varphi) = T(\check{\varphi}) \quad (\text{for } \varphi(\cdot) \in D).$$

It is easily verified that

$$(2.3) \quad [\check{f}]^0 = [f]^0{}^\vee \quad (\text{for } f(\cdot) \in \mathcal{F}).$$

Clearly,

$$(2.4) \quad \check{\check{S}} = S \quad (\text{for } S \text{ in } D').$$

2.5. Lemma. Suppose that $a < 0$. If $\varphi(\cdot) \in D$ then $\varphi < (-\infty, -a)$ if (and only if) $\check{\varphi} < (a, \infty)$.

PROOF. If $\varphi < (-\infty, -a)$ we can use 1.2 to assert that $\varphi(\cdot)$ vanishes outside of some closed subinterval $[-\lambda, -\mu]$ of $(-\infty, -a)$. Therefore, $-\mu < -a$ and $a < \mu$, so that

$$(1) \quad [\mu, \infty) \subset (a, \infty).$$

If $x > \lambda$ and $t < \mu$ then $-x < -\lambda$ and $-\mu < -t$, which implies that $\varphi(-t) = 0 = \varphi(-x)$ (since $\varphi(\cdot)$ vanishes outside of the interval $[-\lambda, -\mu]$); consequently, $\check{\varphi}(\cdot)$ vanishes on the set $(-\infty, \mu) \cup (\lambda, \infty)$: this means that the support of $\check{\varphi}(\cdot)$ is contained in the interval $[\mu, \lambda]$: the conclusion $\check{\varphi} < (a, \infty)$ now comes from (1) and 1.2.

Conversely, suppose that $\check{\varphi} < (a, \infty)$. From 1.2 it follows that $\check{\varphi}(\cdot)$ vanishes outside of some interval $[\mu, \lambda] \subset (a, \infty)$. Therefore, $a < \mu$ and $-\mu < -a$, so that

$$(2) \quad (-\infty, -\mu] \subset (-\infty, -a).$$

If $t > -\mu$ and $x < -\lambda$ then $-t < \mu$ and $-x > \lambda$, which implies that $\check{\varphi}(-t) = \check{\varphi}(-x) = 0$ (since $\check{\varphi}(\cdot)$ vanishes outside of $[\mu, \lambda]$); consequently, $\varphi(\cdot)$ vanishes on the set $(-\infty, -\lambda) \cup (-\mu, \infty)$; therefore, the support of $\varphi(\cdot)$ is contained in $[-\lambda, -\mu]$: the conclusion $\varphi < (-\infty, -a)$ is now immediate from (2).

2.6. Theorem. Suppose that $a < 0$. If T is a distribution, then

$$\bigcirc(T) \supset (a, \infty) \quad \text{if (and only if)} \quad \bigcirc(\check{T}) \supset (-\infty, -a).$$

PROOF. If $\bigcirc(T) \supset (a, \infty)$ then

$$(1) \quad T(\varphi_1) = 0 \quad (\text{for } \varphi_1 < (a, \infty)).$$

If $\varphi < (-\infty, -a)$ then $\check{\varphi} < (a, \infty)$ (by 2.5), so that $T(\check{\varphi}) = 0$ (by (1)); consequently, $\check{T}(\varphi) = 0$ (by 2.2). We have just seen that $\check{T}(\varphi) = 0$ whenever $\varphi < (-\infty, -a)$: this means that $\bigcirc(\check{T}) \supset (-\infty, -a)$.

Conversely, suppose that $\bigcirc(\check{T}) \supset (-\infty, -a)$: if $\varphi < (-\infty, -a)$ then $\check{T}(\varphi) = 0$. Thus, by 2.2:

$$(2) \quad T(\check{\varphi}) = 0 \quad (\text{for } \varphi < (-\infty, -a)).$$

If $\varphi_1 < (a, \infty)$ we set $\varphi(\cdot) = \check{\varphi}_1(\cdot)$; then

$$(3) \quad \varphi_1(\cdot) = \check{\varphi}(\cdot)$$

and $\check{\varphi} < (a, \infty)$, whence $\varphi < (-\infty, -a)$ (by 2.5), and the equations

$$0 = T(\check{\varphi}) = T(\varphi_1)$$

come directly from (2) and (3). We have just seen that $T(\varphi_1) = 0$ whenever $\varphi_1 < (a, \infty)$: this means that $\bigcirc(T) \supset (a, \infty)$.

2.7. Lemma. If S and T are distributions such that

$$\bigcirc(S) \supset (-\infty, \sigma) \quad \text{and} \quad \bigcirc(T) \supset (-\infty, \tau),$$

then $S * T$ is a distribution such that

$$(2.8) \quad \bigcirc(S * T) \supset (-\infty, \sigma + \tau);$$

moreover,

$$(2.9) \quad S * T = T * S.$$

PROOF. Note that both S and T belong to D'_+ (see [8, p. 172]); note also that $\bigcirc(F)$ is the set-theoretic complement of $\text{Supp } F$. Conclusions 2.8 and 2.9 now follows from Théorème XIII in [8, p. 172].

2.10. Definition. If F_1 and F_2 are distributions such that

$$(2.11) \quad \bigcirc(F_k) \supset (a_k, \infty) \quad (\text{for } k = 1, 2),$$

we set

$$(2.12) \quad F_1 * F_2 = (\check{F} * \check{F}_2)^\vee.$$

2.13. Lemma. *If F_1 and F_2 are distributions satisfying 2.11, then*

$$(2.14) \quad \circ(F_1 * F_2) \supset (a_1 + a_2, \infty).$$

If $f()$ and $g()$ are in $\mathcal{F}_-$, then the equation

$$f * g(t) = \int_{-\infty}^{\infty} f(t-u)g(u) du \quad (\text{for } t \in \mathbb{R})$$

*defines a function $f * g()$ in $\mathcal{F}_-$ such that*

$$(2.15) \quad [f]^0 * [g]^0 = [f * g]^0.$$

PROOF. It is not hard to verify that $*$ is the convolution product as defined in [10, pp. 123—124]; in consequence, 2.14 can be derived from Theorem 5.4—2 in [10, p. 125]; further, 2.15 is proved in [10, pp. 126—127]. The present lemma can be proved directly from 2.12; for example, to establish 2.14, observe that $\circ(\tilde{F}_k)$ contains the interval $(-\infty, -a_k)$ (by 2.11 and 2.6), so that we may use 2.8 to assert that

$$\circ(\tilde{F}_1 * \tilde{F}_2) \supset (-\infty, -a_1 - a_2):$$

conclusion 2.14 now results from one more application of 2.6.

2.16. Lemma. *If $L \in (\mathcal{L})$ and $\circ(R) \supset (0, \infty)$ then $L * R$ and $R * L$ both belong to $(\mathcal{L})$.*

PROOF. From 1.16 it follows the existence of a number $a < 0$ such that $\circ(L) \supset (a, \infty)$; we may therefore apply 2.14 to obtain

$$\circ(L * R) \supset (a + 0, \infty),$$

whence the conclusion $L * R \in (\mathcal{L})$ now comes from 1.16; on the other hand, the conclusion $R * L = L * R$ comes from 2.12 and 2.9.

2.17. Definition. *If F and G are distributions, we set*

$$(2.18) \quad F \otimes G = -F_- * G_- + F_+ * G_+.$$

2.19. Theorem. *If F and G belong to $\mathfrak{B}$, then*

$$(2.20) \quad F \otimes G \text{ belongs to } \mathfrak{B},$$

$$(2.21) \quad (F \otimes G)_- = -F_- * G_- ,$$

and

$$(2.22) \quad (F \otimes G)_+ = F_+ * G_+ ;$$

moreover,

$$(2.23) \quad F \in \mathfrak{B}_+ \text{ implies } F \otimes G = F_+ * G_+ \in \mathfrak{B}_+$$

and

$$(2.24) \quad F \in (\mathcal{L}) \text{ implies } F \otimes G \in (\mathcal{L}).$$

PROOF. Clearly,

$$(1) \quad F \otimes G = A + B, \quad \text{where} \quad B = F_+ * G_+$$

and

$$(2) \quad A = -F_- * G_-.$$

A distribution Q belongs to $\mathfrak{B}_+$ if (and only if) $\circ(Q)$ includes the interval $(-\infty, 0)$: see 1.13 and 1.5. Since F_+ and G_+ belong to $\mathfrak{B}_+$, we can use 2.7 to assert that

$$(3) \quad F_+ * G_+ \quad \text{belongs to} \quad \mathfrak{B}_+.$$

In view of (1)–(3), the conclusion $F \otimes G \in \mathfrak{B}$ can be obtained by proving that

$$(4) \quad F_- * G_- \quad \text{belongs to} \quad \mathfrak{B}_-.$$

Let us prove (4). From 1.18 we see that both F_- and G_- belong to $(\mathcal{L}) + [\mathcal{F}_-]^0$; therefore, the equations

$$F_- = L^F + [f]^0 \quad \text{and} \quad G_- = L^G + [g]^0$$

hold for $L^F \in (\mathcal{L})$, $L^G \in (\mathcal{L})$, $f(\cdot) \in \mathcal{F}_-$, and $g(\cdot) \in \mathcal{F}_-$. Consequently,

$$(5) \quad F_- * G_- = L^F * L^G + L^F * [g]^0 + [f]^0 * L^G + [f]^0 * [g]^0.$$

The three first terms on the right-hand side of (5) are of the form $R * L$ (or $L * R$), where $\circ(R) \supset (0, \infty)$ and $L \in (\mathcal{L})$; in view of 2.16, their sum is an element L_1 of $(\mathcal{L})$:

$$(6) \quad F_- * G_- = L_1 + [f]^0 * [g]^0.$$

From (6) and 2.15 it follows that $F_- * G_- = L_1 + [f * g]^0$, where $f * g(\cdot) \in \mathcal{F}_-$. Therefore, $F_- * G_-$ belongs to the space $(\mathcal{L}) + [F_-]^0$: Conclusion (4) now comes from 1.18.

Having thus proved (4), Conclusions 2.20–2.22 follow directly from (1)–(4) and 1.20. It remains to prove 2.23–2.24. If $F \in \mathfrak{B}_+$ then $F_- = \mathbf{0}$ (by 1.21); consequently, $(F \otimes G)_- = \mathbf{0}$ (by 2.21), which implies $F \otimes G \in \mathfrak{B}_+$ (by 1.21) and $F \otimes G = F_+ * G_+$ (by 2.18). Finally, let us prove 2.24. If $F \in (\mathcal{L})$ then $F \in \mathfrak{B}_-$ (by 1.18), so that $F_+ = \mathbf{0}$ and $F = F_-$ (by 1.22): from 2.18 it therefore follows that

$$(7) \quad F \otimes G = -F * G_-.$$

Our conclusion $F \otimes G \in (\mathcal{L})$ is now obtained by setting $F = L$ and $G_- = R$ in 2.16 (note that $\circ(G_-) \supset (0, \infty)$ since $G_- \in \mathfrak{B}_-$ and 1.12).

2.25. Notations. As indicated at the beginning of this section, the derivative of a distribution F is denoted ∂F (see [4, p. 323]); the Dirac distribution is denoted δ (it is defined by the equation $\delta(\varphi) = \varphi(0)$: see [4, p. 314]).

2.26. Theorem. *If R, S, T belong to $\mathfrak{B}$, then*

$$(2.27) \quad R \otimes S = S \otimes R,$$

$$(2.28) \quad R \otimes (S \otimes T) = (R \otimes S) \otimes T,$$

$$(2.29) \quad \delta \otimes S = S_+,$$

$$(2.30) \quad \partial(R \otimes S) = -\partial R_- * S_- + \partial R_+ * S_+,$$

and

$$(2.31) \quad \partial([1]^0 \otimes S) = S.$$

PROOF. Note that $F \in B_+$ if (and only if) F is a distribution whose support is included in the half-open interval $[0, \infty)$ (see 1.13): we may therefore combine Remark 3 in [4, p. 385] with [4, p. 390] and [4, p. 392] to infer that

$$(1) \quad R_+ * S_+ = S_+ * R_+,$$

$$(2) \quad R_+ * (S_+ * T_+) = (R_+ * S_+) * T_+,$$

and

$$(3) \quad \partial(R_+ * S_+) = \partial R_+ * S_+.$$

Since $\circ(F_-) \supset (0, \infty)$ whenever $F \in \mathfrak{B}$, the corresponding equations

$$(i) \quad R_- * S_- = S_- * R_-,$$

$$(ii) \quad R_- * (S_- * T_-) = (R_- * S_-) * T_-,$$

and

$$(iii) \quad \partial(R_- * S_-) = \partial R_- * S_-$$

can be obtained from (1)–(3) by applying 2.12, 2.16, and 2.13 (alternatively, (i)–(iii) can be obtained by verifying that they are consequences of [4, p. 390] and [4, p. 392]). The equation

$$R \otimes S = -S_- * R_- + S_+ * R_+$$

is from 2.18, (1), and (i); another application of 2.18 now gives 2.27. Next, Definition 2.18 gives

$$R \otimes (S \otimes T) = -R_- * (S \otimes T)_- + R_+ * (S \otimes T)_+,$$

so that, by 2.21–2.22:

$$R \otimes (S \otimes T) = R_- * (S_- * T_-) + R_+ * (S_+ * T_+);$$

we may now apply (2) and (ii) to obtain

$$R \otimes (S \otimes T) = (R_- * S_-) * T_- + (R_+ * S_+) * T_+;$$

but 2.21–2.22 then give

$$R \otimes (S \otimes T) = -(R \otimes S)_- * T_- + (R \otimes S)_+ * T_+,$$

and conclusion 2.28 is now immediate from 2.18.

Next, observe that δ belongs to $\mathfrak{B}_+$; therefore, $\delta_- = 0$ and $\delta_+ = \delta$; the equations

$$\delta \otimes S = \delta_+ * S_+ = \delta * S_+ = S_+$$

are from 2.18, from $\delta_+ = \delta$, and from Proposition 9 in [4, p. 391]. We still have to prove 2.30—2.31. From 2.18 it follows immediately that

$$\partial(R \otimes S) = -\partial(R_- * S_-) + \partial(R_+ * S_+):$$

Conclusions 2.30 is now immediate from (3) and (iii); on the other hand, the equations

$$(4) \quad \partial([1]^0 \otimes S) = -\partial[1]_-^0 * S + \partial[1]_+^0 * S_+ = \delta * S_- + \delta * S_+$$

are from 2.30 and 2.1; on the other hand, the equations

$$(5) \quad \delta * S_- + \delta * S_+ = S_- + S_+ = S$$

are from Proposition 9 in [4, p. 391] and 1.20. Conclusion 2.31 comes directly from (4)—(5).

2.32. Theorem. *If $f()$ and $g()$ belong to $\mathcal{F}$, then $[f]^0 \otimes [g]^0 = [f \wedge g]^0$, where $f \wedge g()$ is the function in $\mathcal{F}$ defined by*

$$(2.33) \quad f \wedge g(t) = \int_0^t f(t-u)g(u) du \quad (\text{for } t \in \mathbf{R}).$$

PROOF. From 2.18, 1.28, and 2.15 it follows that $[f]^0 \otimes [g]^0 = [h]^0$, where

$$(6) \quad h() = -f_- * g_-() + f_+ * g_+();$$

the proof will therefore be completed by showing that $h() = f \wedge g()$. To begin with, suppose that $F() \in \mathcal{F}$ and note that the equations

$$(2.34) \quad F_-(t) = \begin{cases} F(t) & \text{for } t < 0 \\ 0 & \text{for } t \geq 0 \end{cases}$$

and

$$(2.35) \quad F_+(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ F(t) & \text{for } t > 0 \end{cases}$$

are immediate consequences of 1.24—1.26. Next, it is not hard to verify that $f_+ * g_+() = 0$ on $(-\infty, 0)$; therefore, (6) gives

$$(7) \quad h(t) = -f_- * g_-(t) \quad (\text{for } t < 0).$$

If $t < 0$ then

$$(8) \quad f_- * g_-(t) = \int_{-\infty}^0 f_-(t-u)g(u) du = \int_t^0 f(t-u)g(u) du;$$

the first equation is from 2.34 (with $F=g$) and the second is obtained by observing that $u > t$ implies $t-u < 0$, whence $f_-(t-u) = f(t-u)$ (by 2.34 with $F=f$), whereas $u < t$ implies $t-u > 0$ and $f_-(t-u) = 0$. From (8) and 2.33 it follows that

$$-f_- * g_-(t) = f \wedge g(t) \quad (\text{for } t < 0);$$

combining with (7): $h() = f \wedge g()$ on $(-\infty, 0)$. Next, to prove the same relation on $(0, \infty)$, take $t > 0$: the equations

$$(9) \quad f_+ * g_+(t) = \int_0^\infty f_+(t-u)g(u) du = \int_0^t f(t-u)g(u) du$$

come from 2.35 (with $F=g$) and from the fact that $f_+(t-u)=0$ for $u > t$ (see 2.35 with $F=f$). From (9) and 2.33 it follows that

$$(10) \quad f_+ * g_+(t) = f \wedge g(t) \quad (\text{for } t > 0).$$

It is not hard to verify that $f_- * g_-(t) = 0$ (for $t > 0$); consequently, (6) gives

$$h(t) = f_+ * g_+(t) \quad (\text{for } t > 0).$$

Combining with (10), we obtain: $h() = f \wedge g()$ on $(0, \infty)$; since we have already verified that $h() = f \wedge g()$ on $(-\infty, 0)$, we have concluded the proof.

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