

A sub class of α -spiral functions

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1. Introduction

Let $G(\alpha)$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular and analytic in the unit disc $E = \{z: |z| < 1\}$ with the property

$$(1) \quad \operatorname{Re} \left[e^{i\alpha} \frac{zf'(z)}{f(z)} \right] > 0, \quad z \in E \quad \text{and} \quad |\alpha| \leq \frac{\pi}{2}.$$

Functions satisfying conditions (1) are called α -spiral functions and it was shown by ŠPAČEK [3] that such functions are univalent in E . When $|\alpha| = \pi/2$, the function $f(z)$ reduces to the identity function z and hence we exclude such a case from our consideration.

In this paper we shall consider a sub-class $H(\alpha)$ of those functions in $G(\alpha)$ which satisfy the condition

$$(2) \quad \left| e^{i\alpha} \frac{zf'(z)}{f(z)} - (1 + i \sin \alpha) \right| < 1, \quad z \in E, \quad |\alpha| < \frac{\pi}{2}.$$

The class $H(0)$ coincides with the class of functions studied by R. SINGH in [2]. Hence on letting $\alpha=0$ in the theorems proved below, we shall get the results obtained earlier by R. Singh in [2].

2. A representation formula

Theorem 2.1. *Every function $f(z) \in H(\alpha)$ can be represented in the form*

$$(3) \quad f(z) = z \exp \left[e^{-i\alpha} \cos \alpha (2 - \cos \alpha) \int_0^z \frac{\psi(\zeta)}{1 - (1 - \cos \alpha) \psi(\zeta)} \cdot \frac{d\zeta}{\zeta} \right]$$

where $\psi(\zeta)$ is a function satisfying Schwarz's Lemma [I, p. 165].

PROOF. Let

$$(4) \quad \varphi(z) = e^{i\alpha} \frac{zf'(z)}{f(z)} - 1 - i \sin \alpha,$$

then

$$(5) \quad \varphi(0) = \cos \alpha - 1.$$

Let

$$(6) \quad \psi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \varphi(0)\varphi(z)},$$

then $\psi(0)=0$ and $|\psi(z)| < 1$, therefore $\psi(z)$ satisfies the conditions of Schwarz's Lemma.

From (4), (5) and (6) we obtain

$$(7) \quad \frac{zf'(z)}{f(z)} = \frac{1 + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)\psi(z)}{1 - (1 - \cos \alpha)\psi(z)}.$$

From (7) we obtain after some simple computation and integration formula (3).

It is easy to see that every function $f(z)$ represented by formula (3) will belong to the class $H(\alpha)$.

3. Radius of starlikeness

Theorem 3.1. *If $f(z) \in H(\alpha)$ and $|z|=r$, $0 < r < 1$, then*

$$(8) \quad \frac{1 - \cos \alpha(2 - \cos \alpha)r + (1 - \cos \alpha)(\cos 2\alpha + \sin^2 \alpha \cos \alpha)r^2}{1 - (1 - \cos \alpha)^2 r^2} \equiv \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] \equiv \\ \equiv \frac{1 + \cos \alpha(2 - \cos \alpha)r + (1 - \cos \alpha)(\cos 2\alpha + \sin^2 \alpha \cos \alpha)r^2}{1 - (1 - \cos \alpha)^2 r^2}.$$

These bounds are sharp.

PROOF. The right hand side of (7) is subordinate to the function

$$\frac{1 + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)z}{1 - (1 - \cos \alpha)z}$$

which for any r , $0 < r < 1$, maps the disc $|z| \leq r$ onto the disc with centre

$$\frac{1 + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)(1 - \cos \alpha)r^2}{1 - (1 - \cos \alpha)^2 r^2}$$

and radius

$$\frac{\cos \alpha(2 - \cos \alpha)r}{1 - (1 - \cos \alpha)^2 r^2}.$$

Therefore, we obtain

$$(9) \quad \left| \frac{zf'(z)}{f(z)} - \frac{1 + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)(1 - \cos \alpha)r^2}{1 - (1 - \cos \alpha)^2 r^2} \right| \equiv \frac{\cos \alpha(2 - \cos \alpha)r}{1 - (1 - \cos \alpha)^2 r^2}.$$

Inequalities in (8) now follow at once from (9).

The sharp bounds in (7) on the left and on the right are attained on taking $\psi(z)=z$ in the function defined by (7), that is in the function.

$$(10) \quad f(z) = z(1 - (1 - \cos \alpha)z)^{\frac{-\cos \alpha(2 - \cos \alpha)e^{-i\alpha}}{1 - \cos \alpha}} \quad \left(\text{where } \frac{f(z)}{z} = 1 \text{ at } z = 0 \right)$$

at the points $re^{i\theta_1}$ and $re^{i\theta_2}$ respectively where

$$(11) \quad \tan \theta_1 = \frac{-(1 - (1 - \cos \alpha)^2 r^2) \sin \alpha}{2(1 - \cos \alpha)r - (1 + (1 - \cos \alpha)^2 r^2) \cos \alpha},$$

and

$$(12) \quad \tan \theta_2 = \frac{(1 - (1 - \cos \alpha)^2 r^2) \sin \alpha}{2(1 - \cos \alpha)r + (1 + (1 - \cos \alpha)^2 r^2) \cos \alpha}, \quad |z| = r.$$

Theorem 3.2. If $f(z) \in H(\alpha)$ then $f(z)$ is starlike in $|z| < r_0$ where r_0 is the smaller of the two positive roots of the equation

$$(13) \quad 1 - \cos \alpha(2 - \cos \alpha)r + (1 - \cos \alpha)(\cos 2\alpha + \sin^2 \alpha \cos \alpha)r^2 = 0.$$

The result is sharp.

PROOF. The theorem follows immediately from the left hand inequality in (8).

4. Bounds for $|f(z)|$

Theorem 4.1. If $f(z) \in H(\alpha)$ and $|z|=r$, $0 < r < 1$, then

$$(14) \quad r \left(\frac{1 + Ar}{1 - Ar} \right)^{-B/2A} \cdot \frac{1}{(1 - A^2 r^2)^{\cos \alpha/2A}} \cong \\ \cong |f(z)| \cong r \left(\frac{1 + Ar}{1 - Ar} \right)^{B/2A} \cdot \frac{1}{(1 - A^2 r^2)^{\cos \alpha/2A}}$$

where $A = 1 - \cos \alpha$, $B = \cos \alpha(2 - \cos \alpha) > 0$ for $|\alpha| < \pi/2$.

These bounds are sharp only when $\alpha=0$.

PROOF. From (7) we have

$$(15) \quad z \frac{d}{dz} \left(\log \frac{f(z)}{z} \right) = \frac{e^{-i\alpha} B \psi(z)}{1 - A \varphi(z)},$$

or

$$(16) \quad f(z) = z \exp \left[e^{-i\alpha} B \int_0^z \frac{\psi(\zeta)}{1 - A \psi(\zeta)} \cdot \frac{d\zeta}{\zeta} \right].$$

Therefore,

$$(17) \quad |f(z)| = |z| \cdot \exp \left[B \int_0^r \operatorname{Re} \left(\frac{e^{-i\alpha} \psi(te^{i\theta})}{1 - A \psi(te^{i\theta})} \right) \cdot \frac{dt}{t} \right].$$

Since $\frac{\psi(z)}{1-A\psi(z)}$ is subordinate to $\frac{z}{1-Az}$ which for any $r, 0 < r < 1$ maps the disc $|z| \leq r$ onto the disc with centre $\frac{Ar^2}{1-A^2r^2}$ and radius $\frac{r}{1-A^2r^2}$ therefore we have

$$(18) \quad \left| \frac{\psi(z)}{1-A\psi(z)} - \frac{Ar^2}{1-A^2r^2} \right| \leq \frac{r}{1-A^2r^2},$$

which yields

$$(19) \quad \frac{A \cos \alpha r^2 - r}{1-A^2r^2} \leq \operatorname{Re} \left[\frac{e^{-i\alpha} \psi(z)}{1-A\psi(z)} \right] \leq \frac{A \cos \alpha \cdot r^2 + r}{1-A^2r^2}.$$

(17) together with right hand side of (19) yields

$$(20) \quad \begin{aligned} |f(z)| &\leq r \exp \left[B \int_0^r \frac{A \cos \alpha \cdot t + 1}{1-A^2t^2} dt \right] = \\ &= r \exp \left[\frac{B}{2A} \log \frac{1+Ar}{1-Ar} + \frac{\cos \alpha}{2A} \log \frac{1}{1-A^2r^2} \right] = \\ &= r \cdot \left(\frac{1+Ar}{1-Ar} \right)^{B/2A} \cdot \frac{1}{(1-A^2r^2)^{(\cos \alpha)/2A}}. \end{aligned}$$

Similarly we obtain the lower bound for $|f(z)|$.

Remark 1. Equalities on either side of (14) are attained at a point z , $|z|=r$ if and only if $\psi(z)=\varepsilon z$, $|\varepsilon|=1$. But, a computation shows that equalities will occur on either side of (19) at points $re^{i\theta_1}$ and $re^{i\theta_2}$, θ_1, θ_2 being functions of r . Consequently, the estimates made in (20) are not sharp in general because the integration is performed along a ray from the origin.

5. Coefficient estimates

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H(\alpha)$, then

$$(21) \quad |a_2| \leq \cos \alpha (2 - \cos \alpha), \quad |\alpha| < \pi/2,$$

$$(22) \quad |a_n| \leq \frac{\cos \alpha (2 - \cos \alpha)}{n-1} \quad \text{for } n \geq 2 \quad \text{and} \quad \cos \alpha \leq \frac{1}{3}, \quad \alpha = 0.$$

These bounds are sharp.

PROOF. Substituting the power series for $f(z)$ in (7) we get after some simplification

$$\sum_{k=2}^{\infty} (k-1) a_k z^k = \left[\sum_{k=1}^{\infty} (k(1 - \cos \alpha) + e^{-i\alpha} (e^{-i\alpha} + i \sin \alpha \cos \alpha)) a_k z^k \right] \psi(z).$$

Rewriting it in the form

$$\sum_{k=2}^n (k-1)a_k z^k + \sum_{k=n+1}^{\infty} c_k z^k = \left[\sum_{k=1}^{n-1} (k(1-\cos \alpha) + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)) a_k z^k \right] \psi(z),$$

$\sum_{k=n+1}^{\infty} c_k z^k$ being absolutely and uniformly convergent on compact subsets of E . Putting $z = re^{i\theta}$ and using the fact that $\psi(z)$ is bounded by 1, we have

$$(23) \quad \begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=2}^n (k-1)a_k r^k e^{i\theta k} + \sum_{k=n+1}^{\infty} c_k r^k e^{i\theta k} \right|^2 d\theta \leq \\ & \equiv \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{n-1} (k(1-\cos \alpha) + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)) a_k r^k e^{i\theta k} \right|^2 d\theta. \end{aligned}$$

Applying Parseval's identity [1, p. 100] to (23) we get

$$(24) \quad \begin{aligned} & \sum_{k=2}^n (k-1)^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k} \leq \\ & \equiv \sum_{k=1}^{n-1} |k(1-\cos \alpha) + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)|^2 |a_k|^2 r^{2k}. \end{aligned}$$

Since the infinite series is non-negative and $0 < r < 1$, we can write

$$(25) \quad \begin{aligned} & \sum_{k=2}^n (k-1)^2 |a_k|^2 \leq \sum_{k=1}^{n-1} |k(1-\cos \alpha) + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)|^2 |a_k|^2, \\ \text{or} \\ & (n-1)^2 |a_n|^2 \leq |1-\cos \alpha + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)|^2 + \\ & + \sum_{k=2}^{n-1} [|k(1-\cos \alpha) + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha)|^2 - (k-1)^2] |a_k|^2 = \\ & = \cos^2 \alpha (2 - \cos \alpha)^2 - \cos \alpha (2 - \cos \alpha) \sum_{k=2}^{n-1} [k^2 - 2k(\sin^2 \alpha + \cos \alpha) + \sin^2 \alpha] |a_k|^2. \end{aligned}$$

An easy calculation shows that $k^2 - 2k(\sin^2 \alpha + \cos \alpha) + \sin^2 \alpha$ is an increasing function of k and remains positive for $k \geq 3$, $|\alpha| < \pi/2$. For $k=2$, $k^2 - 2k(\sin^2 \alpha + \cos \alpha) + \sin^2 \alpha = (1-\cos \alpha)(1-3\cos \alpha)$ which is non-negative for $\alpha=0$ or for $\cos \alpha \leq 1/3$. Hence for $\alpha=0$ or for $\cos \alpha \leq 1/3$ we obtain from (25), $(n-1)^2 |a_n|^2 \leq \cos^2 \alpha (2 - \cos \alpha)^2$ which yields (22).

Also from (25) we obtain for $|\alpha| < \pi/2$

$$|a_2| \leq \cos \alpha (2 - \cos \alpha)$$

which is (21).

For $1 > \cos \alpha > 1/3$ and $n \geq 3$ on using (21) we obtain from (25)

$$(n-1)^2 |a_n|^2 \leq \cos^2 \alpha (2 - \cos \alpha)^2 + \cos \alpha (2 - \cos \alpha) (1 - \cos \alpha) (3 \cos \alpha - 1) |a_2|^2 \leq \\ \leq \cos^2 \alpha (2 - \cos \alpha)^2 [1 + \cos \alpha (2 - \cos \alpha) (1 - \cos \alpha) (3 \cos \alpha - 1)]$$

or

$$(26) \quad |a_n| \leq \frac{\cos \alpha (2 - \cos \alpha) \sqrt{1 + \cos \alpha (2 - \cos \alpha) (1 - \cos \alpha) (3 \cos \alpha - 1)}}{n-1} \quad \text{for } n \geq 3.$$

For any fixed $n \geq 2$ and for $\alpha = 0$ or $\cos \alpha \leq 1/3$, the function $f(z)$ defined by

$$\frac{zf'(z)}{f(z)} = \frac{1 + e^{-i\alpha}(e^{-i\alpha} + i \sin \alpha \cos \alpha) z^{n-1}}{1 - (1 - \cos \alpha) z^{n-1}}$$

gives sharp estimates.

Remark 2. When $1/3 < \cos \alpha < 1$, $n \geq 3$, we have not been able so far to obtain sharp bounds for $|a_n|$.

References

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